$|G_1 \oplus G_2 \oplus \cdots \oplus G_m| = |G_1| \cdot |G_2| \cdots |G_m|$

Lecture-35

29/10/22

External Direct Product :-

Let $G_1, G_2, \ldots, G_m$ be a finite collection of groups then external direct product of $G_1, G_2, \ldots, G_m$ written as —

$$G_1 \oplus G_2 \oplus G_3 \oplus \cdots \oplus G_m$$
$$= \{(g_1, g_2, g_3, \ldots, g_n) ; g_i \in G_i, i = 1, 2, \ldots, m\}$$

$$G_1 \oplus G_2 = \{(g_1, g_2) ; g_1 \in G_1, g_2 \in G_2\}$$

Eg → $G_1 \to Z_2$ $\quad G_2 \to Z_3$

$$\underset{\{0,1\}}{Z_2} \oplus \underset{\{0,1,2\}}{Z_3} = \begin{Bmatrix} (0,0), (0,1), (0,2) \\ (1,0), (1,1), (1,2) \end{Bmatrix}$$

$$\boxed{(g_1, g_2) * (g_1', g_2') = (g_1 g_1', g_2 g_2')}$$

Ex ① → $U(8) \oplus U(10)$

$U(8) = \{1, 3, 5, 7\}$ $\qquad |U(8)| = 4$

$U(10) = \{1, 3, 7, 9\}$ $\qquad |U(10)| = 4$

$$U(8) \oplus U(10) = \begin{Bmatrix} (1,1) & (1,3) & (1,7), & (1,9) \\ (3,1) & (3,3) & (3,7), & (3,9) \\ (5,1) & (5,3) & (5,7) & (5,9) \\ (7,1) & (7,3), & (7,7), & (7,9) \end{Bmatrix}$$

$$(3,7) * (7,1) = (3 \cdot_8 7, 7 \cdot_{10} 1)$$
$$= (5,7) \in U(8) \oplus U(10)$$

H.W.

Q. Prove that external direct product $G_1 \oplus G_2 \oplus \ldots \oplus G_n$ is a group under operation $*$ defined as —

$$(g_1, g_2) * (g_1', g_2') = (g_1 g_1', g_2 g_2')$$

Eg-①

(a) Does $U(8) \oplus U(10)$ a cyclic group?

(b) Is $U(8) \oplus U(10)$ an abelian group?

Theorem :- If $G_1, G_2 \ldots G_n$ are abelian groups then their external direct product $G_1 \oplus G_2 \oplus \ldots \oplus G_n$ is also abelian group.

Order of an element of a Direct Product

The order of an element of a direct product of a finite no. of finite gps is the least common multiple of the orders of the components of the elements.

$$O(g_1, g_2, \ldots g_n) = \text{lcm}(|g_1|, |g_2|, \ldots |g_n|)$$

where,

$g_1 \in G_1$, $g_2 \in G_2$, ... $g_n \in G_n$

Q. $G = Z_2 \oplus Z_3$ ① Find order of each element in G.
② Is $Z_2 \oplus Z_3$ a cyclic gp?
③ Is G an abelian gp?

$$Z_2 \oplus Z_3 = \left\{ \begin{matrix} (0,0),(0,1),(0,2) \\ (1,0),(1,1),(1,2) \end{matrix} \right\}$$

$|G| = |Z_2| \cdot |Z_3|$

$|G| = 2 \times 3 = 6$

$(0,0)$ $|(0,0)| = \text{lcm}(\underset{Z_2}{|0|}, \underset{Z_3}{|0|})$
$= \text{lcm}(1,1) = 1$ Ans

$|(0,1)| = \text{lcm}(|0|, |1|)$
$= \text{lcm}(1,3)$
$= 3$ Ans

$|(0,2)| = \text{lcm}(|0|, |2|)$
$= \text{lcm}(1,3) = 3$ Ans

$|(1,0)| = \text{lcm}(|1|, |0|)$
$= \text{lcm}(2,1) = 2$ Ans

$|(1,1)| = \text{lcm}(|1|, |1|)$
$= \text{lcm}(2,3) = 6$ Ans

$|(1,2)| = \text{lcm}(|1|, |2|)$
$= \text{lcm}(2,3) = 6$ Ans

Generators $\rightarrow (1,1), (1,2)$

$(1,1) * (1,2) = (0,0)$

$\Rightarrow (1,1)^{-1} = (1,2)$ & $(1,2)^{-1} = (1,1)$

(b) We obtain 2 generators
$\Rightarrow Z_2 \oplus Z_3$ is cyclic group.

(c) G is cyclic gp
$\Longrightarrow G$ is abelian gp.

Exercise

Q. If $G = Z_{25} \oplus Z_5$
Determine whether G is cyclic gp.
If yes, then find all the generators of group G.
Determine the no. of elements of order 5 in G.