~~Cauchy Sequence:- A s~~

Nested Intervals:- A sequence of sets $\{A_n\}$ is called a nested sequence if $A_1 \supset A_2 \supset A_3 \supset \ldots A_n \supset A_{n+1} \supset \ldots$

Nested Interval Theorem:- For each $n \in N$, let $I_n = [a_n, b_n]$ be a non-empty closed and bounded interval on R such that $\{I_n\}$ is a nested sequence with $\lim_{n\to\infty}(\text{length of } I_n) = \lim_{n\to\infty}(b_n - a_n) = 0$

then $\bigcap_{n=1}^{\infty} I_n$ contains exactly one point.

Proof → let $\{I_n\}$ be a nested sequence, then

$I_n \supset I_{n+1} \quad \forall\ n \in N$

i.e. $[a_n, b_n] \supset [a_{n+1}, b_{n+1}] \quad \forall\ n \in N$ —— ①

$\Rightarrow a_n \le a_{n+1} \le b_{n+1} \le b_n \quad \forall\ n \in N$

$\Rightarrow \{a_n\}$ is monotonically increasing sequence bounded above by b_1 and $\{b_n\}$ is monotonically decreasing sequence bounded below by a_1.

$\Rightarrow \{a_n\}$ & $\{b_n\}$ are convergent.

Now, $\lim(b_n - a_n) = 0 \Rightarrow \lim b_n = \lim a_n = l$ (let)

$\because$ a monotonically increasing bounded seq. converges to its supremum.

$\Rightarrow l$ is the supremum of the sequence $\{a_n\}$

Similarly, a monotonically decreasing bounded sequence converges to its infimum

$\Rightarrow l$ is the infimum of the sequence $\langle b_n \rangle$

$\Rightarrow a_n \leq l \leq b_n$

Hence, $l \in I_n$

$\Rightarrow l \in \bigcap_{n=1}^{\infty} I_n$

Now, we have to show that the uniqueness let $l \neq l'$ can belong to $\bigcap_{n=1}^{\infty} I_n$

$\Rightarrow 0 \leq |l' - l| \leq |b_n - a_n|$

$\Rightarrow |l' - l| = 0$

$\Rightarrow \boxed{l = l'}$

$\Rightarrow \lim (b_n - a_n) \to 0$

—— Proved

<u>Cauchy Sequence</u> :- A sequence of $\langle x_n \rangle$ is said to be a cauchy sequence if for each positive number ϵ however small, $\exists$ a positive integer 'n_0' such that ——

$$n, m \geq n_0 \Rightarrow |x_n - x_m| < \epsilon$$

or

$$n \geq n_0 \ \& \ p > 0 \Rightarrow |x_{n+p} - x_n| < \epsilon$$