7/10/22

Theorem:- Every convergent sequence is a cauchy sequence.

Proof → Let $\langle x_n \rangle$ be a convergent sequence and converges to 'l'.
If $\epsilon > 0$ is given, $\exists$ a positive integer n_0 such that —

$$n \geq n_0 \Rightarrow |x_n - l| < \frac{\epsilon}{2}$$

and equally $m \geq n_0 \Rightarrow |x_m - l| < \frac{\epsilon}{2}$

Thus, if $n, m \geq n_0$, then —

$$|x_n - x_m| = |(x_n - l) - (x_m - l)|$$
$$|x_n - x_m| \leq |x_n - l| + |x_m - l|$$
$$|x_n - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$|x_n - x_m| < \epsilon$$

Hence, every convergent sequence is a cauchy sequence.

Proved

Theorem:- Every cauchy sequence is bounded.

Proof → Let $\langle x_n \rangle$ be a cauchy sequence.
Choose $\epsilon = 1$, then $\exists$ a positive integer n_0 s.t. —

$$n, m \geq n_0 \Rightarrow |x_n - x_m| < 1$$

In particular taking $m = n_0 + 1$, we have —

$$|x_n| - |x_{n_0+1}| \leq |x_n - x_{n_0+1}| < 1 \quad \forall\ n \geq n_0$$
$$|x_n| < 1 + |x_{n_0+1}| = \lambda \text{ (say)} \quad \forall\ n \geq n_0 \quad \text{--- (1)}$$

let $M = \max\{|x_1|, |x_2|, \ldots, |x_{n_0-1}|, \lambda\}$
~~Then~~
Evidently,

$$|x_n| \leq M \quad \forall \; n = 1, 2, \ldots, n_0 - 1 \quad \text{--- (2)}$$

and also from eq-(1) →

$$|x_n| \leq M \quad \forall \; n \geq n_0 \quad \text{--- (3)}$$

On combining eq-(2) & (3), we get →

$$|x_n| \leq M \quad \forall \; n \in N$$

Hence, every cauchy sequence is bounded.

Proved

Theorem Every cauchy sequence in R is convergent.

Proof →

let $\langle x_n \rangle$ be a cauchy sequence.
Since every cauchy sequence is bounded so is $\langle x_n \rangle$.
Further since every bounded sequence has a convergent subsequence.
let $\langle y_n \rangle$ be a convergent subsequence of $\langle x_n \rangle$ converges to l.
We have to show that $\langle x_n \rangle \to l$

let $\epsilon > 0$ be given. Since $\langle x_n \rangle$ is a cauchy sequence.
By definition,

$$n, m \geq m_0 \Rightarrow |x_n - x_m| < \frac{\epsilon}{2} \quad \text{--- (1)}$$

Further, since $\langle y_n \rangle$ converges to l,
$\exists$ a positive integer k_0 such that —

$$n \geq k_0 \Rightarrow |y_n - l| < \frac{\epsilon}{2} \quad \text{—— (2)}$$

let $n_0 = \max\{m_0, k_0\}$ so that eq-(1) hold $\forall\, n, m \geq n_0$
and eq-(2) hold $\forall\, n \geq n_0$

Now, since $\langle y_n \rangle$ is a subsequence of $\langle x_n \rangle$
we have —

$y_{n_0} = x_m$ for some $m \geq n_0$

Therefore, for all $n \geq n_0$. Using eq-(1) & (2)
with n n_0, we have —

$|x_n - l| = |(x_n - x_m) + (y_{n_0} - l)|$ since $x_m = y_{n_0}$ for some $m \geq n_0$

$|x_n - l| \leq |x_n - x_m| + |y_{n_0} - l|$

$|x_n - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$

$|x_n - l| < \epsilon \quad \forall\, n \geq n_0$

Hence, every cauchy sequence in R
is convergent.

—— Proved