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Lecture-37

Q Find the order of subgroup $H = \langle 5 \rangle \oplus \langle 3 \rangle$ in the group $G = Z_{30} \oplus Z_{12}$

Solⁿ → $|H| = |\langle 5 \rangle| \cdot |\langle 3 \rangle|$

$\quad\quad\quad\quad \downarrow Z_{30} \quad\quad \downarrow Z_{10}$

$\langle 5 \rangle = \{5, 10, 15, 20, 25, 0\}$

$|\langle 5 \rangle| = 6$

$\langle 3 \rangle = \{3, 6, 9, 0\}$

$|\langle 3 \rangle| = 4$

$|H| = 6 \times 4 = 24$ Ans

Q Determine the no. of elements of order 15 and the no. of cyclic subgroups of order 15 in $G = Z_{30} \oplus Z_{20}$

$Z_{30} \rightarrow |a| = 1, 2, 3, 5, 6, 10, 15, 30$

$Z_{20} \rightarrow |b| = 1, 2, 4, 5, 10, 20$

$G = Z_{30} \oplus Z_{20}$

Order $15 = |(a,b)| = \text{lcm}(|a|, |b|)$

Case I → $\text{lcm}(15, 1) = 15$

$\phi(15) \cdot \phi(1) = 15\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = 8$

Case II → $\text{lcm}(3, 5) = 15$

$\phi(3) \cdot \phi(5) = 2 \times 4 = 8$

Case III → $\text{lcm}(15, 5) = 15$

$\phi(15) \cdot \phi(5) = 8 \times 4 = 32$

Total no. of elements $= 48$ Ans

No. of cyclic subgroups in $Z_{30} \oplus Z_{20}$ of order $15 = \frac{48}{\phi(15)} = \frac{48}{8} = 6$ Ans

Exercise ① Show that $G \oplus H$ is abelian iff G & H are abelian.

Exercise ② Show that $G = Z_2 \oplus Z_2 \oplus Z_2$ has total seven subgroups of order 2.

Q. Find the order of each element in $Z_2 \oplus Z_4$.

Solⁿ → $Z_2 = \{0, 1\}$

$Z_4 = \{0, 1, 2, 3\}$

$G = Z_2 \oplus Z_4$

$$Z_2 \oplus Z_4 = \left\{ \begin{matrix} (0,0), (0,1), (0,2), (0,3) \\ (1,0), (1,1), (1,2), (1,3) \end{matrix} \right\}$$

$|(0,0)| = \text{lcm}(1,1) = 1$

$|(0,1)| = \text{lcm}(1,4) = 4$

$|(0,2)| = \text{lcm}(1,2) = 2$

$|(0,3)| = \text{lcm}(1,4) = 4$

$|(1,0)| = \text{lcm}(2,1) = 2$

$|(1,1)| = \text{lcm}(2,4) = 4$

$|(1,2)| = \text{lcm}(2,2) = 2$

$|(1,3)| = \text{lcm}(2,4) = 4$

Exercise-③

How many subgroups of order 4 $Z_2 \oplus Z_4$

Theorem:- Let G & H be finite cyclic groups then direct product $G \oplus H$ is cyclic iff $|G|$ & $|H|$ are relatively co-prime.

Proof.→ Given→ G, H are cyclic groups & $G \oplus H$ is also cyclic

Suppose $|G| = m$

$|H| = n$

TPT→ $\gcd(m, n) = 1$

$|G| = |\langle g \rangle| = m \Rightarrow g^m = e$

$|H| = |\langle h \rangle| = n \Rightarrow h^n = e$

Suppose $\gcd(m, n) \neq 1$

$\Rightarrow \gcd(m, n) = t$ (say)

$\Rightarrow m = k_1 t, \quad n = k_2 t \quad (k_1, k_2 \in \mathbb{Z})$

$\frac{m}{t} = k_1, \quad \frac{n}{t} = k_2$

$|g^{m/t}| = ?$

$(g^{m/t})^t = g^m = e \Rightarrow |g^{m/t}| = t$

Similarly, $|h^{n/t}| = ?$

$(h^{n/t})^t = h^n = e \Rightarrow |h^{n/t}| = t$

P.T.O.

★ Fundamental Theorem of Cyclic Groups:-

- Every subgroup of cyclic group is cyclic.
- $|G| = n$, then order of its subgroup divides n.
- In addition, for each +ve divisor 'k' of n, the group G has exactly one subgroup of order 'k'.

Given → $G \oplus H$ is cyclic

$(g^{m/t}, e)$, $(e, h^{n/t})$

$H_1 = \langle (g^{m/t}, e) \rangle$, $|H_1| = \text{lcm}(|g^{m/t}|, |e|) = t$

$H_2 = \langle (e, h^{n/t}) \rangle$, $|H_2| = \text{lcm}(|e|, |h^{n/t}|) = t$

$|H_1| = |H_2| = t$

We have 2 cyclic subgroups of $G \oplus H$ which is the contradiction.

(From theorem → for each +ve divisor 'k' of n, the grp G has exactly one subgroup of order k)

$\Rightarrow \gcd(m, n) = 1$ —— Proved

Converse → Suppose $\gcd(m, n) = 1$

TPT → $G \oplus H$ is cyclic

Let $(a, b) \in G \oplus H$

s.t. $G \oplus H = \langle (a, b) \rangle$

or $|G \oplus H| = |(a, b)| = \text{lcm}(|a|, |b|)$

$= \text{lcm}(m, n) = mn$

$\Rightarrow |G \oplus H| = |(a, b)| = mn$

$\Rightarrow G \oplus H$ is cyclic

—— Proved