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Cauchy's General Principle of Convergence

A sequence of real numbers is convergent iff it is a cauchy sequence.

Q. If $\langle a_n \rangle$ is a sequence defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

Find → $\lim_{n\to\infty} |a_{n+1} - a_n|$. Does the sequence satisfy cauchy criterion?

We have, $a_{n+1} - a_n = \frac{1}{n+1}$

$$\lim_{n\to\infty} |a_{n+1} - a_n| = \lim_{n\to\infty} \left(\frac{1}{n+1}\right) = 0 \quad \text{Ans}$$

Now, let $\epsilon = 1/2$. If $\langle a_n \rangle$ satisfy cauchy criterian, we have a positive integer 'm' such that —

$$n \geq m \text{ and } p > 0 \Rightarrow |a_{n+p} - a_n| < \frac{1}{2} \quad \text{①}$$

let us choose $n = m$, $p = m$ then —

$$|a_{n+p} - a_n| = |a_{2m} - a_m| = \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{2m}$$

$$|a_{2m} - a_m| > \frac{1}{2m} + \frac{1}{2m} + \dots + \frac{1}{2m}$$

$$|a_{2m} - a_m| > \frac{1}{2}$$

Thus, ∃ no +ve integer m for which eq ① is satisfie[d]
Hence, cauchy criterion is not satisfied.

Q. Show that the $\langle x_n \rangle$ defined by —
$x_1 = 1$, $x_2 = 2$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for $n > 2$
is convergent. Find its limit.

The given recursion formula can be written as —

$$x_n - x_{n-1} = \frac{1}{2}(x_{n-1} + x_{n-2}) - x_{n-1}$$

$$x_n - x_{n-1} = \frac{-1}{2}(x_{n-1} - x_{n-2})$$

Replacing n by $(n+1)$, we get $\longrightarrow$

$$x_{n+1} - x_n = \frac{-1}{2}(x_n - x_{n-1})$$

and so on.

$$x_{n+1} - x_n = \left(\frac{-1}{2}\right)^2 (x_{n-1} - x_{n-2}) = \left(\frac{-1}{2}\right)^3 (x_{n-2} - x_{n-3})$$

$$= \ldots\ldots = \left(\frac{-1}{2}\right)^{n-1}(x_2 - x_1) = \left(\frac{-1}{2}\right)^{n-1}$$

Thus,

$$|x_{n+1} - x_n| = \left(\frac{1}{2}\right)^{n-1}$$

Now, if $m > n$, we have —

$$|x_n - x_m| = |(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \ldots\ldots + (x_{m-1} - x_m)|$$

$$|x_n - x_m| \le |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \ldots\ldots + |x_{m-1} - x_m|$$