Convergence & Divergence of Series

Series → An expression of the form $u_1 + u_2 + u_3 + \dots + u_n + \dots$ in which the successive terms are formed according to some definite laws is called series. It is briefly denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$.

Alternating Series: A series in which terms are alternatively +ve and -ve is called alternating series.

Eg → $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

Convergent Series: An infinite series $u_1 + u_2 + u_3 + \dots + u_n + \dots$ is said to be convergent if its n^{th} partial sum (S_n) tends to a finite limit say 'S' as n tends to infinity. Here, S is called the sum of the series.

i.e.

$$\lim_{n \to \infty} S_n = S$$

or

$$|S_n - S| < \epsilon \quad \forall \; n \geq n_0$$

Divergent Series:- If $\lim S_n = \infty$ or $(-\infty)$ then the given series is divergent.

Oscillatory Series:- If $\lim S_n$ is not unique then the given series is oscillatory.

Theorem:- If a series $\sum u_n$ is convergent then $\lim u_n = 0$. but converse need not be true.

Proof → let S_n be the sum of first n terms of the given series.
Since, the series is convergent.
$\Rightarrow \lim S_n = S$ (finite quantity)

Now, $u_n = S_n - S_{n-1}$
$\Rightarrow \lim u_n = \lim S_n - \lim S_{n-1}$
$\lim u_n = S - S$
$\lim u_n = 0$

——— Proved

Note:- If $\lim u_n \neq 0$, then the series $\sum u_n$ is not convergent.

Counter eg:- Consider series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots \frac{1}{\sqrt{n}} + \dots$

$u_n = \frac{1}{\sqrt{n}}$ $\quad \lim u_n = 0$

$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots \frac{1}{\sqrt{n}}$

$S_n > \frac{n}{\sqrt{n}} = \sqrt{n} \to \infty$

which shows that the series is divergent but $\lim u_n = 0$.