Archimedian Property of Real Numbers:

Theorem → "If x and y are any two positive real numbers with $y<x$, then there exists a positive integer 'n' such that $ny > x$"

Proof → Let if possible $ny \leq x \ \forall \ n \in N$ then ~~then~~ the set $S = \{ny : n \in N\}$ is non-empty and bounded above by x.

By the completeness property in R, $\sup S$ exists in R. Let it be α.

Now, we have $ny \leq \alpha \ \forall \ n \in N$

$$\Rightarrow (n+1)y \leq \alpha \ \forall \ n \in N$$

$$\Rightarrow ny \leq \alpha - y \ \forall \ n \in N$$

$\Rightarrow (\alpha - y)$ is an upper bound of S and $(\alpha - y) < \alpha$

But it is contrary to the fact that α is the $\sup S$.

Therefore, our assumption that $ny \leq x \ \forall \ n \in N$ is wrong.

Hence, $ny > x \ \forall \ n \in N$

Proved

<u>Theorem</u> If $x \in R$, then $\exists$ a rational number 'n' such that $x < n$.

<u>Proof</u> → We suppose that if possible that $x \geq n$

$\Rightarrow n \leq x \;\; \forall \; n \in N$

$\Rightarrow$ The set N is bounded above

By Completeness Property of real numbers the set N must has supremum say 'u'.

$Sup\, N = u$

$u - 1 < u$

Therefore, $(u-1)$ is not an upper bound of N

$\Rightarrow \exists$ atleast one element 'm' such that

$m > u - 1$

$\Rightarrow m + 1 > u$

Now, $m \in N \Rightarrow (m+1) \in N$

Thus, $(m+1) > u \;\; \forall \; (m+1) \in N$

This inequality contradicts the fact that u is the $Sup\, N$.

Hence, our assumption is wrong.

Therefore, $\exists$ at least one natural number 'n' such that $x < n$

—— Proved

Theorem For any real number 'x', there exist two integers m and n such that $m < x < n$.

Proof $\rightarrow$ By previous result, we have $x < n$ —— (1)

Further, $-x$ is also a real number
$\exists$ a +ve integer $p > -x$ i.e. $-p < x$

Putting $-p = m$, we have $m < x$ —— (2)

On combining eq (1) & (2), we get $\rightarrow$
$$m < x < n \quad \forall \; m, n \in I$$