p-Series: The series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$

where p is some real number, is
- convergent if $p > 1$ &
- divergent if $p \leq 1$

Proof→ We consider the following cases —

Case I→ When $p > 1$

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \dots$$

$$\sum \frac{1}{n^p} < 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \dots$$

$$\sum \frac{1}{n^p} < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \dots$$

$$\sum \frac{1}{n^p} < \frac{1}{\left(1 - \frac{1}{2^{p-1}}\right)} \to S$$

which is a finite quantity.
Hence, the given series is convergent.

Case II→ When $p = 1$

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$

$$\sum \frac{1}{n} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots$$

$$\sum \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

Since, $u_n = \frac{1}{2}$ and $\lim u_n = \frac{1}{2} \neq 0$

Hence, the given series is divergent for $p = 1$

Case III → When $p < 1$

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$$

$$\sum \frac{1}{n^p} > 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

$$\sum \frac{1}{n^p} > \sum \frac{1}{n}$$, which is divergent (from case II)

Hence, the given series is also divergent.

Proved

Imp **Leibnitz's Test (Test for alternating convergent series):-**

If the terms of an infinite series are —

(i) alternating +ve and -ve

(ii) numerically in decreasing order

(iii) $\lim u_n = 0$

then the series is convergent.

Imp **Test for Alternating oscillatory series:-**

If the terms of an infinite series are alternating +ve and -ve and $\lim u_n \neq 0$ then the series is oscillatory.