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Demorgan's Test:-

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n > 1, \text{ convergent}$$
$$< 1, \text{ divergent}$$
$$= 1, \text{ (test fails)}$$

Q Test for convergence of the series →

$$\frac{1^2}{2^2}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}+\ldots\ldots$$

Sol^n → $u_n = \dfrac{1^2\cdot 3^2\cdot 5^2\ldots(2n-1)^2}{2^2\cdot 4^2\cdot 6^2\ldots(2n)^2}$

$$u_{n+1} = \frac{1^2\cdot 3^2\cdot 5^2\ldots\ldots(2n+1)^2}{2^2\cdot 4^2\cdot 6^2\ldots(2n+2)^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{4n^2+8n+4}{4n^2+4n+1}$$

$$\lim \frac{u_n}{u_{n+1}} = 1 \quad \text{(Test fails)}$$

$$\lim\left(\frac{u_n}{u_{n+1}}-1\right)n = \lim \frac{4n^2+3n}{4n^2+4n+1}$$
$$= 1 \quad \text{(Test fails)}$$

$$\lim\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = \lim\left[\frac{(-n-1)\log n}{4n^2+4n+1}\right]$$
$$= \lim\frac{(-n^2-n)}{4n^2+4n+1}\cdot\lim\frac{\log n}{n}$$
$$= \frac{-1}{4}\times 0 = 0 < 1$$

Hence, by Demorgan's Test, the given series is divergent.

Ratio Test

Higher Ratio Test → Demorgan's Test

Logarithmic Test → Second logarithmic test

Logarithmic Test :-

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} > 1, \text{ (convergent)}$$
$$< 1, \text{ (divergent)}$$
$$= 1, \text{ (Test fails)}$$

Second Logarithmic Test :-

$$\lim \left(n \log \frac{u_n}{u_{n+1}} - 1 \right) \log n > 1, \text{ (convergent)}$$
$$< 1, \text{ (divergent)}$$
$$= 1, \text{ (Test fails)}$$

Q Test the convergence of the series →

$$1 + \frac{2x}{2!} + \frac{3^2x^2}{3!} + \frac{4^3x^3}{4!} + \ldots\ldots, \quad x > 0$$

Sol^n →
$$u_n = \frac{n^{n-1}}{n!} x^{n-1}$$

$$u_{n+1} = \frac{(n+1)^n x^n}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \left(\frac{n}{n+1}\right)^{n-1} \cdot \frac{1}{x}$$

$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{1}{\left(1+\frac{1}{n}\right)^{n-1}} \cdot \frac{1}{x}$$

$$= \lim \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x}$$

$$= \frac{1}{ex}$$

When $\frac{1}{ex} > 1$, then series is convergent

When $\frac{1}{ex} < 1$, then series is divergent.

When $ex = 1 \Rightarrow x = \frac{1}{e}$

$$\frac{u_n}{u_{n+1}} = \left(\frac{n}{n+1}\right)^{n-1} \cdot e$$

$$\log \frac{u_n}{u_{n+1}} = (n-1) \log\left(\frac{n}{n+1}\right) + \log e$$

$$\lim n \log \frac{u_n}{u_{n+1}} = \lim \left[n \left\{(n-1) \log \frac{1}{\left(1+\frac{1}{n}\right)} + 1\right\}\right]$$

$$= \lim n \left[-(n-1) \log\left(1+\frac{1}{n}\right) + 1\right]$$

$$= \lim n \left[-(n-1) \left\{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots\right\} + 1\right]$$

$$= \lim n \left[\frac{3}{2n} - \frac{5}{6n^2} + \cdots\right]$$

$$= \frac{3}{2} > 1$$

Hence, by logarithmic Test, the given series is convergent at $ex = 1$