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Q Test for convergence:→

$$1+\frac{\alpha\cdot\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2+\cdots$$

$$u_n=\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{1\cdot2\cdots n\,\gamma(\gamma+1)\cdots(\gamma+n-1)}$$

$$u_{n+1}=\frac{\alpha(\alpha+1)\cdots(\alpha+n)\beta(\beta+1)\cdots(\beta+n)}{1\cdot2\cdots(n+1)\,\gamma(\gamma+1)\cdots(\gamma+n)}$$

$$\lim\frac{u_n}{u_{n+1}}=\lim\frac{n^2+(1+\gamma)n+\gamma}{n^2+(\alpha+\beta)n+\alpha\beta}\cdot\frac{1}{x}=\frac{1}{x}$$

If $\frac{1}{x}>1$, then series is convergent.

If $\frac{1}{x}<1$, then series is divergent.

If $x=1$ then test fails.

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim\left[\frac{(1+\gamma-\alpha-\beta)n^2+(\gamma-\alpha\beta)n}{n^2+(\alpha+\beta)n+\alpha\beta}\right]$$
$$=1+\gamma-\alpha-\beta$$

If $1+\gamma-\alpha-\beta>1$, then series is convergent.
If $1+\gamma-\alpha-\beta<1$, then series is divergent.

Test fails when $1+\gamma-\alpha-\beta=1$
$$\gamma=\alpha+\beta$$

$$\lim_{n\to\infty} \log n \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 \right\} = \lim \log n \frac{(-\alpha\beta n - \alpha\beta)}{n^2 + (\alpha+\beta)n + \alpha\beta}$$

$$= \lim \frac{\left(\frac{\log n}{n}\right)(-\alpha\beta)\left(1+\frac{1}{n}\right)}{1 + (\alpha+\beta)\frac{1}{n} + \frac{\alpha\beta}{n^2}}$$

$$= 0 \times (-\alpha\beta)$$

$$= 0 < 1$$

Hence, by Demorgan's Test the given series is divergent when $\alpha + \beta = \gamma$.

Q Examine the convergence or divergence of the series $\rightarrow$

$$x + x^{1+\frac{1}{2}} + x^{1+\frac{1}{2}+\frac{1}{3}} + x^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} + \cdots$$

Sol^n $\rightarrow$ $u_n = x^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}$

$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{1}{x^{1/n+1}}$$

$$= 1 \text{ (Test fails)}$$

$$\lim \left(n \log \frac{u_n}{u_{n+1}}\right) = \lim n\left(0 - \log x^{1/n+1}\right)$$

$$= \lim (-n) \frac{1}{n+1} \log x$$

$$= \lim \frac{1}{\left(1+\frac{1}{n}\right)} \log \frac{1}{x}$$

$$= \log \frac{1}{x}$$

If $\log\frac{1}{x} > 1$, then series is convergent.

If $\log\frac{1}{x} < 1$, then series is divergent.

Test fails when $\log\frac{1}{x} = 1 = \log e$

$$\frac{1}{x} = e \quad \text{or} \quad x = \frac{1}{e}$$

$$n\log\frac{u_n}{u_{n+1}} = \frac{n}{n+1}$$

$$\lim \log n\left(n\log\frac{u_n}{u_{n+1}} - 1\right) = \lim \log n\left(\frac{n}{n+1} - 1\right)$$

$$= \lim\left[\log n\left(\frac{-1}{n+1}\right)\right]$$

$$= \lim \frac{\log n}{n}\frac{(-1)}{\left(1+\frac{1}{n}\right)}$$

$$= 0 < 1$$

Hence, by IInd logarithmic Test, the given series is divergent when $x = \frac{1}{e}$.