Theorem: For any real number 'x', there exist two integers m and n such that $m < x < n$.

Proof → By previous result, we have $x < n$ —— ①

Further, $-x$ is also a real number
$\exists$ a +ve integer $p > -x$ i.e. $-p < x$

Putting $-p = m$, we have $m < x$ —— ②

On combining eq ① & ②, we get →
$m < x < n \quad \forall \; m, n \in \mathbb{Z}$

—— Proved

18/8/22

Theorem: For any real number 'x' there exist a unique integer 'n' such that $n \leq x < n+1$

Proof → Put $[x] = n$ where $[x]$ denote the greatest integer of x
Obviously, $n \leq x$ —— ①

Further to prove that $x < n+1$
let if possible $x \geq n+1$

Since $n+1$ is an integer it would follow that $[x] \geq n+1$ i.e. $n \geq n+1$, which is not possible
Hence, $x < n+1$ —— ②
from eq ① & ②, we get →
$n \leq x < n+1$

—— Proved

Theorem: Between any two distinct rational number there lies atleast one rational number and hence infinitely many rational numbers.

Proof: → Let x and y be any two distinct rational numbers such that — $x < y$

Now, $x < y$

$\Rightarrow x + x < x + y$

$\Rightarrow x < \frac{x+y}{2}$ —— ①

Now, $x < y \Rightarrow x + y < y + y$

$\frac{x+y}{2} < y$ —— ②

from eq-① & ②, we get →

$x < \frac{x+y}{2} < y$

$x < r < y$ where $r = \frac{x+y}{2}$

Clearly, r is a rational number and it lies between x & y

Repeat the above argument for x & r, y and r we get rational numbers r_1 and r_2 s.t. —

$x < r_1 < r < r_2 < y$

Continuing in this way, definitely we can find infinitely many rational numbers between x and y.

—— Proved