Imp Q. If the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$

is rearranged in such a way the limit of ratio of number of +ve terms to number of −ve terms tend to k, then the sum of rearranged series is $\frac{1}{2}\log 4k$.

Solⁿ → The given series is —

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots \quad \text{——— (1)}$$

is convergent and its sum is $\log 2$

Suppose $\frac{m}{n}=k$

$$\underbrace{\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2m-1}\right)}_{+ve}-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots\frac{1}{2n}\right)+\ldots \quad \text{——— (2)}$$

The sum of first m terms and 'n' negative terms of (2) is ——

$$\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2m-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)$$

$$1+\frac{1}{2}+\cdots+\frac{1}{n}=\log n+\gamma_n \qquad \gamma_n\to\gamma \text{ as } n\to\infty$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2m-1}+\frac{1}{2m}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2m}\right)$$

$$-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2m}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2m}\right)$$

$$-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$

$$=\log 2m+\gamma_{2m}-\frac{1}{2}(\log m+\gamma_m)-\frac{1}{2}(\log n+\gamma_n)$$

$$=\frac{1}{2}\log\left(\frac{4m^2}{mn}\right)+\gamma_{2m}-\frac{1}{2}\gamma_m-\frac{1}{2}\gamma_n$$

$$=\frac{1}{2}\log 4\left(\frac{m}{n}\right)+\underset{\downarrow\,\gamma}{\gamma_{2m}}-\frac{1}{2}\underset{\gamma}{\gamma_m}-\frac{1}{2}\underset{\gamma}{\gamma_n}$$

$$=\frac{1}{2}\log 4k \text{ (in limiting case.)}$$

Eg $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}\cdots$

$$=\frac{1}{2}\log\frac{4m}{n} \qquad \begin{array}{l} m\to \text{no of +ve terms} \\ n\to \text{no. of -ve terms}\end{array}$$

$$=\frac{1}{2}\log\left(4\times\frac{1}{2}\right)$$

$$=\frac{1}{2}\log 2 \quad \underline{\text{Ans}}$$