Take a +ve integer ν' greater than N such that $M_{\nu'}$ is the supremum of $\alpha_{\nu'}, \alpha_{\nu'+1}, \alpha_{\nu'+2}, \ldots$

Then there exist another number α_{N_1} such that

$$\alpha_{N_1} > M_{\nu'} - \varepsilon$$

$$\alpha_{N_1} > \Delta - \varepsilon \quad \text{———} (v)$$

Thus, we have shown that associate a number Δ with the sequence of $\{\alpha_n\}$ such that eq (i) for $\varepsilon > 0$, $\exists$ a +ive integer ν such that

$$\alpha_n < \Delta + \varepsilon, \quad \forall\ n \geq \nu$$

There are infinitely mambers of $\{\alpha_n\}$ which exceed $\Delta - \varepsilon$

then this number Δ is called limit superior of $\{\alpha_n\}$.

we write, $\overline{\text{limit}}$ sup $\{\alpha_n\}$

or $\overline{\text{limit}}\ \{\alpha_n\}$

Similarly, we can shown that associate a number λ with the sequence of $\{\alpha_n\}$ such that

for $\varepsilon > 0$, $\exists$ a +ive integer ν such that

$$\alpha_n > \lambda - \varepsilon, \quad \forall\ n \geq \nu$$

There are infinately many members of $\{\alpha_n\}$ which is less than $\lambda + \varepsilon$

this number λ is called limit inferior of $\{\alpha_n\}$, we write, $\underline{\text{lim}}$ inf $\{\alpha_n\}$

or $\underline{\text{lim}}\ \{\alpha_n\}$

Clearly, $\Delta \geq \lambda$

$\lim x_n = \alpha$

$\overline{\lim}\, x_n = \Delta$

$\underline{\lim}\, x_n = \lambda$

$\Delta = \lambda = \alpha$

Ex → $\{x_n\} = \left\{\frac{1}{n}\right\}$

$= \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \frac{1}{n+1} \ldots\right\}$

$M_n = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}$

$= \sup\left\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \ldots\right\}$

$= \frac{1}{n}$

$\overline{\lim}\, x_n = \lim M_n = \lim \frac{1}{n} = 0$

$m_n = \inf\left\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \ldots\right\}$

$= 0$

$\lim m_n = 0 = \underline{\lim}\, x_n$

Since, $\overline{\lim}\, x_n = \underline{\lim}\, x_n = 0$

Hence $\lim x_n = 0$

Ex → $\{x_n\} = \{n\}$

$M_n = \sup\{n, n+1, n+2, \ldots\} = \infty$

$\overline{\lim}\, x_n = \lim M_n = \infty, \quad \forall n$

$m_n = \inf\{n, n+1, n+2, \ldots\}$

$m_n = n$

$\underline{\lim}\, x_n = \lim m_n = \infty$

since, $\overline{\lim}\, x_n = \underline{\lim}\, x_n = \infty$

Hence $x_n = \infty$