Riemann Integration:

Let f be a bounded function in $[a,b]$

Let D be a division of $[a,b]$

$$D=\{a=x_0<x_1<x_2<\dots<x_{r-1}<x_r<\dots<x_{n-1}<x_n=b\}$$

The interval $[a,b]$ has been divided into 'n' subintervals $[x_0,x_1],[x_1,x_2],\dots[x_{r-1},x_r]\dots[x_{n-1},x_n]$

f is bounded in $[a,b]$

It is bounded in each subinterval $[x_{r-1},x_r]$ $\quad 1\le r\le n$

Let $M_r=\text{Sup} f$ in $[x_{r-1},x_r]$

$m_r=\text{Inf} f$ in $[x_{r-1},x_r]$

$M=\text{Sup} f$ in $[a,b]$

$m=\text{Inf} f$ in $[a,b]$

The length of $[x_{r-1},x_r]=x_r-x_{r-1}=\delta_r$

Upper Sum $=S(D)=M_1\delta_1+M_2\delta_2+\dots+M_r\delta_r+\dots+M_n\delta_n$

$$S=U(P,f)=\sum_{r=1}^{n}M_r\delta_r$$

Lower Sum $=s(D)=m_1\delta_1+m_2\delta_2+\dots+m_r\delta_r+\dots+m_n\delta_n$

$$s=L(P,f)=\sum_{r=1}^{n}m_r\delta_r$$

Interval $[a,b]$ can be divided into 'n' sub-intervals by infinitely many ways. Consequently, a set of upper sums & another set of lower sums are possible.

$$m\le m_r\le M_r\le M$$

or $\quad m\delta_r\le m_r\delta_r\le M_r\delta_r\le M\delta_r$

or $\quad \sum_{r=1}^{n}m\delta_r\le\sum_{r=1}^{n}m_r\delta_r\le\sum_{r=1}^{n}M_r\delta_r\le\sum_{r=1}^{n}M\delta_r$

or $\quad m(b-a)\le s(D)\le S(D)\le M(b-a)$

$=x_1-x_0+x_2-x_1+\dots+x_n-x_{n-1}=(b-a)$

The set of upper sum as well as the set of lower sum are bounded.

☆ The infimum of the set of upper sums is called upper integral of f over [a,b]. It is written as —

$$U[f;[a,b]] = \overline{\int_a^b} f(x)\,dx$$

$$= \text{Inf of set of upper sum}$$

☆ The supremum of set of lower sums is called lower integral of f over [a,b]. It is written as —

$$L[f,[a,b]] = \underline{\int_a^b} f(x)\,dx = \text{Sup of set of lower sum}$$

A bounded f^n 'f' is said to be Riemann integrable over [a,b] if lower integral & upper integral are equal.
The common value of Riemann integral of 'f' over [a,b] we write —

$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$