Theorem :→ If P_1 and P_2 are any two partitions of $[a, b]$. Then $U(P_2, f) \geq L(P_1, f)$

Or

No lower Darboux sum can exceed any Darboux upper sum.

Theorem :→ The lower Riemann integral can not exceed the upper Riemann integral.

Or

$$\underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx$$

Proof → Let $P[a, b]$ denote the set of all partitions of $[a, b]$.

Let $P_1, P_2 \in P$. Since no lower sum can exceed any upper sum, we have

$$L(P_1, f) \leq U(P_2, f) \quad \text{—— ①}$$

eq ① is true for each $P_1 \in P$

Keeping P_2 fixed, we see that the set $\{L(P_1, f) ; P_1 \in P\}$ has an upper bound $U(P_2, f)$. Again we know that

$$\underline{\int_a^b} f\,dx = \sup\{L(P_1, f) ; P_1 \in P\}$$

But, supremum $\leq$ any upper bound.

Hence, we get, $\underline{\int_a^b} f\,dx \leq U(P_2, f)$ —— ②

which is true for each $P_2 \in P$.

Norm of D :- = length of largest sub-interval of $[a, b] = \|D\|$

$\omega(D)$ = Oscillatory sum

$$= \sum_{r=1}^{n} M_r \delta_r - \sum_{r=1}^{n} m_r \delta_r$$

$$= S(D) - s(D)$$

$$= \sum M_r \delta_r - \sum m_r \delta_r$$

$$= \sum_{r=1}^{n} (M_r - m_r) \delta_r$$

$$= \sum_{r=1}^{n} O_r \delta_r$$

where O_r is the oscillations of f in $[x_{r-1}, x_r]$.

Theorem :- Let f be a bounded fn defined on $[a, b]$ and let m & M be the infimum and supremum of f on $[a, b]$, then for any partition P of ~~closed~~ $[a, b]$ we have, $m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$.