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Darboux theorem :- Let f be a bounded fn defined on $[a, b]$. Then for every $\epsilon > 0$, there correspond $\delta > 0$ such that

(i) $U(P, f) < \int_a^b f(x)\,dx + \epsilon$

(ii) $L(P, f) > \int_a^b f\,dx - \epsilon$

for all partition P such that $\|P\| < \delta$, $\|P\|$ being norm of partition P.

Let $P = \{ a = x_0, x_1, x_2, \ldots\ldots x_{r-1}, x_n = b\}$

Then the sum defined by

$$\sum_{r=1}^{n} f(\xi_r) \cdot \delta_r$$

where ξ_r be any arbitrary point $[x_{r-1}, x_r)$

Alternative definition of Integrability :-

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Theorem :- If f is bounded and integrable over $[a, b]$ then to every $\varepsilon > 0$ there corresponds $\delta > 0$, such that for every partition $P = \{a = x_0, x_1, x_2, \ldots x_{r-1}, x_r \ldots x_n = b\}$ of norm $\leq \delta$ (or $\|P\| \leq \delta$) and for every arbitrary choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum f(\xi_r)\, \delta_r - \int_a^b f\, dx \right| < \varepsilon$$

where $\delta_r = x_r - x_{r-1}$

Proof :- Since f is a bounded and integrable f^n, therefore

$$\int_a^{-b} f\, dx = \int_{-a}^{b} f\, dx = \int_a^b f\, dx \quad \text{———} \ (*)$$

Let ε be any positive number, then by Darboux theorem, there exist $\delta > 0$ such that $\|P\| \leq \delta$.

$$U(P, f) < \int_a^{-b} f\, dx + \varepsilon = \int_a^b f\, dx + \varepsilon \quad \text{———} \ (1)$$

$$L(P, f) > \int_{-a}^{b} f\, dx - \varepsilon = \int_a^b f\, dx - \varepsilon \quad \text{———} \ (2)$$

If ξ_r be any point $I_r = [x_{r-1}, x_r]$ of P, we have,

$$L(P, f) \leq \sum_{r=1}^{n} f(\xi_r)\, \delta_r \leq U(P, f) \quad \text{———} \ (3)$$

from (1), (2) & (3), we get

$$\int_a^b f\, dx - \varepsilon < \sum_{r=1}^{n} f(\xi_r)\, \delta_r < \int_a^b f\, dx + \varepsilon$$

or

$$\left| \sum_{r=1}^{n} f(\xi_r)\, \delta_r - \int_a^b f\, dx \right| < \varepsilon$$

Proved