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Necessary and Sufficient Condition for integrability :-

Theorem :- A necessary and sufficient condition for the integrability of a bounded function f is that for every $\varepsilon > 0$, there corresponds $\delta > 0$ such that for every partition P, whose norm is $< \delta$, the oscillatory sum $\omega(P, f)$ is $< \varepsilon$, i.e.

$$U(P, f) - L(P, f) < \varepsilon$$

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Proof :- The bounded fn f being integrable

$$\int_a^{-b} f\,dx = \int_{-a}^{b} f\,dx = \int_a^b f\,dx$$

Let ε be any +ve number, then by Darboux's theorem we have

$$U(P, f) < \int_a^{-b} f\,dx + \frac{\varepsilon}{2} = \int_a^b f\,dx + \frac{\varepsilon}{2}$$

and

$$L(P, f) > \int_a^b f\,dx - \frac{\varepsilon}{2} = \int_a^b f\,dx - \frac{\varepsilon}{2}$$

$$\int_a^b f\,dx - \frac{\varepsilon}{2} < L(P, f) \leq U(P, f) < \int_a^b f\,dx + \frac{\varepsilon}{2}$$

$$\Rightarrow \omega(P, f) = U(P, f) - L(P, f) < \varepsilon$$

Conversely :-

$$\omega(P, f) < \varepsilon$$

$$U(P, f) - L(P, f) < \varepsilon$$

$$\left[U(P, f) - \int_a^{-b} f\,dx\right] + \left[\int_a^{-b} f\,dx - \int_{-a}^{b} f\,dx\right] + \left[\int_{-a}^{b} f\,dx - L(P, f)\right] < \varepsilon$$

Each one of three numbers being non-negative, It follows that $0 < \int_a^{-b} f\,dx - \int_{-a}^{b} f\,dx < \varepsilon$

as ε is an arbitrary positive number, we see that the non-negative number

$\int_a^{-b} f\,dx - \int_{-a}^{b} f\,dx$ is less than every positive number and hence.

$$\int_a^{-b} f\,dx - \int_{-a}^{b} f\,dx = 0$$

$$\Rightarrow \int_a^{-b} f\,dx = \int_{-a}^{b} f\,dx$$

Hence $f \in R$-integrable

Theorem :- Every continuous fn^n is integrable.

Proof :- Suppose that f is continuous fn^n in $[a, b]$.

Since f is continuous, it is bounded. Also it is uniformly continuous.

Let ε be any +ve number.

We divide $[a, b]$ into finite number of sub-intervals say I_r, such that the oscillation of f in each of these subintervals is $< \frac{\varepsilon}{b-a}$

Let δ_r be the length of I_r.

We have for this partition,

$$\omega(P, f) = \sum M_r \delta_r - \sum m_r \delta_r$$

$$= \sum (M_r - m_r)\,\delta_r$$

$$< \sum_{r=1}^{n} \frac{\varepsilon}{b-a}\,\delta_r = \frac{\varepsilon}{b-a} \sum \delta_r = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$

Thus $\omega(P, f) < \varepsilon$, Hence $f \in R$-integrable.