First Mean Value theorem :-

If f is bounded and integrable in $[a, b]$ then there exists a number μ lying b/w the bounds of f such that

$$\int_a^b f\,dx = \mu(b-a)$$

Proof :- Let M, m be bounds of f in $[a, b]$ so that $m \leq \mu \leq M$

$$\Rightarrow m(b-a) \leq \mu(b-a) \leq M(b-a) \quad \text{---} (1)$$

if $b > a$

But, we know that

$$m(b-a) \leq \int_a^b f(x)\,dx \leq m(b-a) \quad \text{---} (2)$$

from (1) & (2), we get $\int_a^b f(x)\,dx = \mu(b-a)$

where $m \leq \mu \leq M$

Proved

Fundamental theorem of integral calculus :-

If f is a bounded integrable in $[a, b]$ then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

Proof :- let ϵ be any +ve number. since $\phi' = f$ is bounded and integrable in $[a, b]$. then there exist a partition

$$P = \{a = x_0, x_1, x_2, \ldots\ldots x_{r-1}, x_r \ldots x_n = b\}$$

such that $\left| \sum_{r=1}^{n} \phi'(\xi_r)\,\delta_r - \int_a^b \phi'(x)\,dx \right| < \epsilon$ --- (1)

we particularize the arbitrary point $\xi_r \in I_r$ in the following manner.

By Lagrang mean value theorem of diff we have

$$\phi(x_r) - \phi(x_{r-1}) = \phi'(\xi_r) \cdot \delta_r$$

$$\Rightarrow \sum_{r=1}^{n} \phi'(\xi_r) \cdot \delta_r = \sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})]$$

$$= \phi(b) - \phi(a) \quad \text{——— (2)}$$

from (1) & (2) we get,

$$\left| \phi(b) - \phi(a) - \int_a^b \phi'(x)\,dx \right| < \epsilon$$

Since ϵ is a small +ive number, so we conclude that $\int_a^b \phi'(x)\,dx = \phi(b) - \phi(a)$

or $$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$ Proved.