Theorem :- If f is monotonic in $[a, b]$ then it is integrable in $[a, b]$.

Proof :- Clearly, f is bounded and $f(a)$ & $f(b)$ are its two bounds.

Let ϵ be any positive number and $P = \{a = x_0, x_1, \ldots \ldots x_n = b\}$ be any partition of $[a, b]$ such that length of each sub-interval is $< \dfrac{\epsilon}{f(b) - f(a) + 1}$

Let $\delta_r = x_r - x_{r-1}$

Here $M_r = f(x_r)$, $m_r = f(x_{r-1})$

$I_r = [x_{r-1}, x_r]$

we have,

$$\omega(P,f) = \sum (M_r - m_r)\delta_r$$
$$= \sum \{f(x_r) - f(x_{r-1})\}\delta_r$$
$$< \frac{\varepsilon}{f(b)-f(a)+1} \sum_{r=1}^{n} \{f(x_r) - f(x_{r-1})\}$$

$$= \frac{\varepsilon}{f(b)-f(a)+1} [f(b) - f(a)] < \varepsilon$$

Hence, f is integrable in [a, b].

Ques:→ Give an example of a Riemann integrable fn on [0, 1] which is not monotonic in [0, 1]

Solun:→ Let $f(x) = |x - \frac{1}{2}|$, $\forall x \in [0, 1]$

Here f(x) is monotonic decreasing in $[0, \frac{1}{2}]$ and monotonic increasing in $[\frac{1}{2}, 1]$.
Hence f is not monotonic in [0, 1].
Since f is continuous in [0, 1], therefore it is integrable.

Ques:→ Show by an example that every bounded fn need not be integrable.

Or

Let f(x) be defined on [a, b] as follows

$$f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$$

Show that f is not integrable on [a, b].

Solun:→ $U(P,f) = \sum_{r=1}^{n} M_r \delta_r$, $L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$= \sum_{r=1}^{n} \delta_r = b - a$, $= 0$

$$\overline{\int_a^b} f\,dx = \lim U(P,f) = b-a$$

$$\underline{\int_a^b} f\,dx = \lim L(P,f) = 0$$

Since $\overline{\int_a^b} f\,dx \neq \underline{\int_a^b} f\,dx$

Hence $f \notin R$

Abel's lemma :-

(i) If $a_1, a_2, \ldots, a_n$ is a monatonically decreasing set of n positive numbers and,

(ii) $v_1, v_2, \ldots v_n$ is a set of any n numbers &

(iii) ~~on~~ k, K are two numbers such that

$k < v_1 + v_2 + \ldots + v_p < K$, for $1 \leq p \leq n$

then $a_1 k < \sum_{r=1}^{n} a_r v_r < a_1 K$.