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Bonnet's Theorem :-

If $\int_a^b f(x)\,dx$ and $\int_a^b \psi(x)\,dx$ both exist, ψ is monotonically decreasing and positive in $[a,b]$ then there exist a point $\xi \in [a,b]$ such that

$$\int_a^b f(x)\,\psi(x)\,dx = \psi(a)\int_a^{\xi} f(x)\,dx.$$

Proof :- Let $P = \{a=x_0, x_1, x_2, \ldots x_{r-1}, x_r, \ldots, x_n = b\}$ be any partition of $[a,b]$. Let M_r & m_r be the bounds of f in $I_r = [x_{r-1}, x_r]$.

Let $\xi_1 = a$ & ξ_r where $r \neq 1$, be any point of I_r.

we have
$$m_r \delta_r \leq \int_{x_{r-1}}^{x_r} f(x)\,dx \leq M_r \delta_r ,$$

and
$$m_r \delta_r \leq f(\xi_r)\,\delta_r \leq M_r \delta_r$$

Putting, $r = 1, 2, 3, \ldots p$, where $p \leq n$
and adding, we obtain
$$\sum_{r=1}^{p} m_r \delta_r \leq \int_a^{x_p} f(x)\,dx \leq \sum_{r=1}^{p} M_r \delta_r$$

and
$$\sum_{r=1}^{p} m_r \delta_r \leq \sum_{r=1}^{p} f(\xi_r)\,\delta_r \leq \sum_{r=1}^{p} M_r \delta_r$$

Thus, we have
$$\left| \int_a^{x_p} f(x)\,dx - \sum_{r=1}^{p} f(\xi_r)\,\delta_r \right| \leq \sum_{r=1}^{p} (M_r - m_r)\,\delta_r$$
$$\leq \sum_{r=1}^{n} (M_r - m_r)\,\delta_r$$
$$= \sum_{r=1}^{n} O_r \delta_r$$

$$\left(\int_a^{x_p} f(x)\,dx - \sum_{r=1}^{n} O_r \delta_r\right) \leq \sum_{r=1}^{p} f(\xi_r)\,\delta_r \leq \int_a^{x_p} f(x)\,dx + \sum_{r=1}^{n} O_r \delta_r$$

Now, $\int_a^{t} f(x)\,dx$, being a continuous f^n with t as variable is bound.
Let C and D be its bounds.
Therefore, we have
$$C - \sum_{r=1}^{n} O_r \delta_r \leq \sum_{r=1}^{p} f(\xi_r)\,\delta_r \leq D + \sum_{r=1}^{n} O_r \delta_r$$

By using Abel's lemma, we have
$V_r = f(\xi_r)\,\delta_r$, $a_r = \psi(\xi_r)$
$a_1 = \psi(\xi_1) = \psi(a)$

$$\psi(a)\left[C-\sum_{r=1}^{n} O_r\delta_r\right] \leq \sum_{r=1}^{n}\psi(\xi_r)\, f(\xi_r)\,\delta_r$$

$$\leq \psi(a)\left[D+\sum_{r=1}^{n} O_r\delta_r\right]$$

Let the norm of the partition tend to 0. then we obtain

$$C\psi(a) \leq \int_a^b f(x)\,\psi(x)\,dx \leq D\psi(a)$$

$$\Rightarrow \int_a^b f(x)\,\psi(x)\,dx = \mu\,\psi(a)$$

where μ is some number b/w C & D

The continuous fn $\int_a^t f(x)dx$ must assume for some $\xi \in [a,b]$. the value μ which lies b/w its bounds C & D. Thus we obtain

$$\int_a^b f(x)\,\psi(x)\,dx = \psi(a)\int_a^{\xi} f(x)\,dx$$

Proved