25/11/22

Second mean value theorem :-

If $\int_a^b f(x)\,dx$ and $\int_a^b \phi(x)\,dx$ both exist and ϕ is monotonic in $[a,b]$, then there exists $\xi \in [a,b]$ such that

$$\int_a^b f(x)\,\phi(x)\,dx = \phi(a)\int_a^{\xi} f(x)\,dx + \phi(b)\int_{\xi}^{b} f(x)\,dx$$

Proof :- Let ϕ be monotonically decreasing so that the function $\psi(x) = \phi(x) - \phi(b)$ is monotonically decreasing and positive.

Therefore, there exist a number $\xi \in [a,b]$ such that

$$\int_a^b f(x)\,[\phi(x)-\phi(b)]\,dx = [\phi(a)-\phi(b)]\int_a^{\xi} f(x)\,dx$$

$$\Rightarrow \int_a^b f(x)\,\phi(x)\,dx = \phi(a)\int_a^{\xi} f(x)\,dx + \phi(b)\left[\int_a^b f(x)\,dx - \int_a^{\xi} f(x)\,dx\right]$$

$$= \phi(a)\int_a^{\xi} f(x)\,dx + \phi(b)\int_{\xi}^{b} f(x)\,dx$$

Proved

Ques :- Prove that the function f defined on $[0,4]$ by $f(x) = [x]$ is integrable in $[0,4]$ and

$$\int_0^4 f(x)\,dx = 6.$$

Solu :- Theorem :- A bounded function f is integrable in $[a,b]$, if the set of its points of discontinuity is finite.

we have, by definition,

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ 3, & 3 \le x < 4 \end{cases}$$

Here $f(x)$ is bounded and has four points of discontinuity at $x = 1, 2, 3, 4$. since the point of discontinuity of f on $[0,4]$ are finite is number. so f is integrable in $[0,4]$.

$$\int_0^4 f(x)\,dx = \int_0^4 [x]\,dx$$

$$= \int_0^1 0\,dx + \int_1^2 dx + \int_2^3 2\,dx + \int_3^4 3\,dx$$

$$= 0 + [x]_1^2 + [2x]_2^3 + [3x]_3^4$$

$$= 1 + 2 + 3 = \underline{6}$$

Proved

Ques: Evaluate $\int_0^2 x[2x]\,dx$.

$= \frac{17}{4}$

Ques: If $f(x) = x$, $\forall x \in [0, 3]$ and $P = \{0, 1, 2, 3\}$ be a partition of $[0, 3]$, then find $L(P, f)$.