Theorem A subset N of $\mathbb{R}$ is a nbd of a point $p \in \mathbb{R}$ iff there exists an open interval (a,b) such that $p \in (a,b) \subset N$

Proof → Suppose N is a nbd of p.
Then, by definition of nbd, $\exists$ a small positive number $\epsilon > 0$ such that —

$$(p-\epsilon, p+\epsilon) \subseteq N$$

Thus, we have —
$p \in (a,b) \subseteq N$ where $a = p-\epsilon$ and $b = p+\epsilon$

Conversely, Suppose, $\exists$ an open interval (a,b) such that —

$$p \in (a,b) \subseteq N$$

Since, $a < p < b$ we have $p-a > 0$ & $p+a > 0$

Let us choose $\epsilon = \min\{p-a, p+a\}$ so that $\epsilon > 0$

Then, as shown in previous theorem, we have — $(p-\epsilon, p+\epsilon) \subseteq (a,b) \subseteq N$

Hence, N is the nbd of p.

— Proved

Theorem: Any superset of a nbd of a point is also a nbd of that point.

Proof → Let M be a nbd of a point p.
So, $\exists$ a small positive number $\epsilon > 0$ s.t. —
$(p-\epsilon, p+\epsilon) \subseteq M$

Now if S is a superset of M i.e. $M \subset S$
it follows that $(p-\epsilon, p+\epsilon) \subseteq S$
Hence, S is also nbd of p.

—— Proved

Theorem: The intersection of two nbds of a point is also a nbd of that point.

Let M and N be two nbds of a point p.
Then, $\exists$ a small positive numbers ϵ_1 & ϵ_2 such that —
$(p-\epsilon_1, p+\epsilon_1) \subseteq M$ & $(p-\epsilon_2, p+\epsilon_2) \subseteq N$

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ so that —
$(p-\epsilon, p+\epsilon) \subseteq (p-\epsilon_1, p+\epsilon_1) \subseteq M$ &
$(p-\epsilon, p+\epsilon) \subseteq (p-\epsilon_2, p+\epsilon_2) \subseteq N$

It follows that —
$(p-\epsilon, p+\epsilon) \subseteq M \cap N$
Hence, $M \cap N$ is also a nbd of p.

—— Proved