Matrix Eigenvalue Problems

Consider a matrix equation

$$A\,C = \lambda C \qquad \text{—(1)}$$

when an eigenvalue equation is presented in this form, we can call it a matrix eigenvalue equation and call the vectors C that solve it eigenvectors. once a matrix eigenvalue problem has been solved, we can recover the eigen functions of the original problem from their expansion;

$$\psi = \sum_i c_i \phi_i$$

A Preliminary Example →

we consider here a simple problem of 2D motion in which a particle slides frictionlessly in an ellipsoidal basin. If we release the particle at an arbitrary point in the basin, it will start to move downhill in the (-ve) gradient direction which in general will not aim directly at the potential minimum at the bottom of the basin. the particle's overall trajectory will then be a complicated path as sketched in figure.

our objective is to find the positions. It will represent simple one-dimensional oscillatory motion.

we consider a potential,

$$V(x,y) = ax^2 + bxy + cy^2$$

a, b, c are parameters in ranges that describes an ellipsoidal basin with a minimum in V at $x=y=0$

Now calculate F_x and F_y

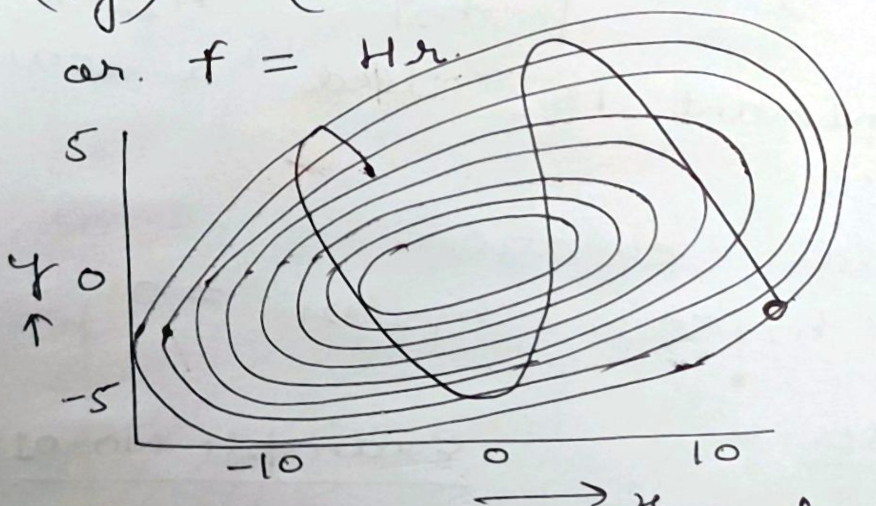
$$F_x = \frac{-\partial V}{\partial x} = -2ax - by$$

$$F_y = \frac{-\partial V}{\partial y} = -bx - 2cy$$

we write the force equation in matrix form,

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} -2a & -b \\ -b & -2c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or. $f = Hr$



trajectory of sliding particle of unit mass starting from rest at (8.0, -1.92)

the condition $\frac{F_x}{F_y} = \frac{x}{y}$ is equivalent to the statement that f and r are proportional So we can write.

$$Hr = \lambda r$$

Here H is a known matrix, while λ and r are to be determined.
this is an eigenvalue equation, and the column vectors r are its solutions are its eigenvectors and corresponding values of λ are its eigenvalues.

$$Hr = \lambda r$$

is a homogeneous linear equation system.

$$(H - \lambda I) r = 0$$

It will have the unique $r = 0$.
unless $|H - \lambda I| r = 0$

$$\det (H - \lambda I) r = 0$$

$$\det (H - \lambda I) = \begin{vmatrix} h_{11} - \lambda & h_{12} \\ h_{21} & h_{22} - \lambda \end{vmatrix} = 0$$

this determinant is called a secular determinant.

the secular equation.

$$(h_{11} - \lambda)(h_{22} - \lambda) - h_{12} h_{21} = 0$$

Some Examples →

Example NO. 01

2D - Ellipsoidal Basin

$$H = \begin{pmatrix} -2 & \sqrt{5} \\ \sqrt{5} & -6 \end{pmatrix}$$

Secular equation

$$\det (H - \lambda I) = \begin{vmatrix} -2-\lambda & \sqrt{5} \\ \sqrt{5} & -6-\lambda \end{vmatrix}$$

$$= \lambda^2 + 8\lambda + 7 = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = -7$$