e.g.

(i) Every open interval / open sphere is open set

(ii) $\mathbb{R}$ and ϕ are open sets

THEOREMS ON OPEN SET

Theorem-1 Show that in any metric space X each open sphere is an open set

Proof. Let $S_r(x_0)$ is an open sphere in X.

Let $x \in S_r(x_0) \Rightarrow d(x, x_0) < r$ — (1)

$\Rightarrow r - d(x, x_0)$ is +ve real no.

Take $r_1 = r - d(x, x_0)$ — (2)

Further let $y \in S_{r_1}(x) \Rightarrow d(y, x) < r_1$ — (3)

Now $d(y, x_0) \leq d(y, x) + d(x, x_0)$

$< r_1 + d(x, x_0)$ by (3)

$\therefore d(y, x_0) < r$ by (2) $\Rightarrow y \in S_r(x_0)$

$\Rightarrow S_{r_1}(x) \subset S_r(x_0)$

Therefore $S_r(x_0)$ is an open set.

Theorem-2 Let X be a metric space. A subset G of X is open $\Leftrightarrow$ it is a union of open spheres

Proof. Let G is open in X.

To prove that G is union of open spheres

If G is empty then G is union of empty class of open spheres. If G is non empty, then since it is open so each of its points is the centre of an open sphere contained in it, and it is the union of all the open spheres contained in it.

Conversely we assume that G is the union of a class S of open spheres. Let $x \in G$

$\Rightarrow \exists$ an open sphere $S_r(x_0)$ in S s.t. $x \in S_r(x_0)$

and also $S_r(x_0) \subset G$ (6)

$\Rightarrow \exists$ an open sphere $S_{r_1}(x)$ s.t. $S_{r_1}(x) \subset S_r(x_0)$

$\Rightarrow S_{r_1}(x) \subset G \quad \because S_r(x_0) \subset G$

$\Rightarrow G$ is open.

Theorem-3 Let (X, d) be a metric space. Then

(i) any union of open sets in X is open set

(ii) any finite intersection of open sets is open

Proof.- (i) Let $\{G_i : i \in \Delta\}$ be an arbitrary class of open sets. Let $G = \bigcup_{i \in \Delta} G_i$

To prove that G is open

Let $x \in G \Rightarrow x \in G_i$ for any i

$\Rightarrow \exists\, S_r(x)$ s.t. $S_r(x) \subset G_i$

$\because G_i$ is open

$\Rightarrow S_r(x) \subset G_i \subset \bigcup G_i = G$

i.e. for $x \in G$ $\exists$ an open sphere $S_r(x) \subset G$

Hence G is open

(ii) Let $\{G_1, G_2, \ldots G_n\}$ be a finite class of open sets in G and let $G = \bigcap_{i=1}^{n} G_i$

To prove that G is open

Let $x \in G \Rightarrow x \in G_i$ for each $i = 1, 2 \ldots n$

$\Rightarrow$ for each i ~~$\exists$ +ve real number r_i s.t. $S_{r_i}(x) \subset G_i$~~ — (1)

Let r is the smallest number in the set $\{r_1\ r_2\ r_3 \ldots r_n\}$ then $S_r(x) \subset S_{r_i}(x)$ for each i so $S_r(x) \subseteq G_i$ for each i by (1)

$\Rightarrow S_r(x) \subset G$. Hence G is open.

Theorem 4. Every non-empty open set on the real line is the union of a countable disjoint class of open intervals

Proof. Let G be a non empty open subset of the real line. Let $x \in G$ and since G is open, x is centre of a bounded open interval contained in G. Let I_x denotes the union of all open intervals which contains x and are contained in G, then

(i) I_x is an open interval which contains x and is contained in G

(ii) I_x contains every open interval which contains x and is contained in G.

(iii) y is another point in I_x then $I_x = I_y$

Further if x and y are distinct points in G then I_x and I_y are either disjoint or identical; for if they have a common point z then $I_x = I_z$ & $I_y = I_z \Rightarrow I_x = I_y$ otherwise they are disjoint. Now define the class I of all disjoint sets of the form I_x for the point x in G. This is the disjoint class of open intervals and G is obviously its union.

Now we have to show that I is countable for which let G_r be set of all rational points in G. G_r is clearly non empty. Define a mapping $f: G_r \rightarrow I$ given as

for each r in G_r, let $f(r)$ be that unique interval in I which contains r so that I is countable since G_r is countable.

LIMIT POINT – Let (X, d) be a metric space. A point $x_0 \in X$ is called a limit point of a set $A \subset X$ if every nbd of x_0 contains at least one point of A other than x_0

e.g. The subset $\{1, 1/2, 1/3 ---\}$ of the real lin

~~Ex~~ for the interval [0,1], the all real numbers x s.t. $0 \le x \le 1$ are limit points
Set of integers has no limit point.
Every real number is a limit point of the set of all rationals.