Theorem — Let X be a complete metric space and let Y is a subspace of X. Then Y is complete $\Leftrightarrow$ it is closed.

Proof — Let Y is complete. To prove that Y is closed. Let y_0 is be limit point of Y. For each positive integer n, $S_{1/n}(y_0)$ contains a point y_n in Y. It is clear that $\langle y_n \rangle$ converges to y_0 in X and is cauchy sequence in Y but Y is complete so $\langle y_n \rangle$ also converges y_0 in Y. Y is therefore closed.

We now assume that Y is closed. We have to prove that Y is complete: Let $\langle y_n \rangle$ is a cauchy sequence in Y. It is also a cauchy sequence in X and X is complete so that $\langle y_n \rangle$ converges to x (say) in X. We show that x is in Y.

If $\langle y_n \rangle$ has only finitely distinct points then x is that point infinitely repeated and is thus in Y. On the other hand if $\langle y_n \rangle$ has infinitely many distinct points, then x is limit point of the set of points of the sequence. It is also a limit point of Y and since Y is closed, x is in Y.

Theorem — **CANTOR INTERSECTION THEOREM**

Let X be a complete metric space and let $\langle F_n \rangle$ is a decreasing sequence of non empty closed subsets of X s.t. $d(F_n) \to 0$. Thus $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Proof — Since $d(F_n) \to 0$, for each $\epsilon > 0$, $\exists$ a +ve integer m_0 s.t. $d(F_{m_0}) < \epsilon$ —— (1)

Again since $\langle F_n \rangle$ is decreasing sequence, we have

$$m, n \geq m_0 \Rightarrow F_m, F_n \subset F_{m_0}$$

$$\Rightarrow x_m, x_n \in F_{m_0} \Rightarrow d(x_n, x_m) < \epsilon \quad [\text{from (1)}]$$

Thus $\langle x_n \rangle$ is a cauchy sequence in X, since X is complete, $x_n \to x_0$ for some $x_0 \in X$.

Now we shall show that

$$x_0 \in \bigcap_{n=1}^{\infty} F_n$$

Let m be +ve integer then

$$n > m \Rightarrow F_n \subset F_m$$

$$\Rightarrow x_n \in F_m \text{ —— (2)}$$

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Since $x_n \to x_0$. The sequence is eventually in every nbd of x_0 and by (2) every nbd of x_0 contains an infinite no. of the points of F_n. Thus x_0 is a limit point of F_n. Since F_n is closed, $x_0 \in F_n$ and also n is arbitrary, we have $x_0 \in \bigcap_{n=1}^{\infty} F_n$

Suppose there is another point $x_0^* \in \cap F_n$ then $d(x_0, x_0^*) \leq d(F_n)$ for every n. Therefore $d(x_0, x_0^*) \rightarrow 0$ as $n \to \infty$. Hence $x_0 = x_0^*$ and so $\cap F_n$ contains exactly one point x_0.

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A is dense ⟺ $\bar{A} = X$ ⟺ the only closed superset of A is X ⟺ the only open set ~~Metric Space~~ disjoint from A is ϕ ⟺ A intersects every non-empty open set ⟺ A intersects every open sphere

DENSENESS

Let (X, d) be metric space and $A \subset X$

A is dense in itself if $A \subset D(A)$

A is said to be dense or every where dense in X if $\bar{A} = X$

A is said to be dense in a set $B \subset X$ if $B \subset \bar{A}$

A is said to be ~~dense~~ some where dense if $(\bar{A})^\circ \neq \phi$ i.e. $\bar{A}$ contains some open sets

A is said to nowhere dense if

$(\bar{A})^\circ = \phi$ ⟺ $\bar{A}$ does not contain any non-empty open set (symbol)

⟺ Each non empty open set has an open sphere disjoint from A

CATAGORY A subset of a metric space is said to be of first catagory iff it can be written as union of countable family of nowhere dense sets otherwise it is said to be of second catagory

Remarks

(i) If A is nowhere dense then $\bar{A}$ is not entire space

(ii) If A is nowhere dense in X then each open sphere contains a closed sphere which contains no point of A.

Bair Catagory Theorem: If $\langle A_n \rangle$ is sequence of nowhere dense sets in a complete metric space X, then ∃ a point in X which is not in any of the A_n's

Proof. Since X is open, A_1 is nowhere dense, ∃ an open sphere S_1 of radius less than 1 which is disjoint from A_1. Let F_1 be concentric closed sphere whose radius is one half that of S_1 and consider its interior $Int.(F_1)$. Since A_2 is nowhere dense, $Int.(F_1)$ contains an open sphere S_2 of radius less than $\frac{1}{2}$ which is disjoint from A_2. Let F_2 be concentric closed sphere whose radius is one half of S_2 and consider its interior. Since A_3 is nowhere dense, $Int(F_2)$ contains an open sphere S_3 of radius less than $\frac{1}{4}$ which which is disjoint from A_3. Continuing in the same manner we get a decreasing sequence $< F_n >$ of the non empty closed spheres (subsets) of X such that $d(F_n) \to 0$. Since X is complete then from Cantor's intersection theorem $\exists$ a point x in X which is all in F_n's. This point is clearly all in S_n's and therefore it is not in any of A_n's (since S_n is disjoint from A_n).