M.Sc. (Math)

TOPOLOGY

MMA-303

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METRIC SPACE

Let X be a non empty set. A real function, called distance function $d: X \times X \to R$ is called a metric on X if

(i) $d(x,y) \geq 0 \quad \forall x,y \in X$

(ii) $d(x,y) = 0 \Leftrightarrow x = y$

(iii) ~~$d(x,y) = d(y,x)$ (Symmetry)~~

(iv) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in X$ (triangle inequality)

i.e. the function d assigns to each pair (x,y) of elements of X, a non negative real number $d(x,y)$ & $d(x,y)$ is called distance b/w x and y

The non empty set X together with a metric d is called a metric space and then it is denoted by (X,d)

It often happens that several metrics can be defined on a single given set and distinct metrics make the set into different metric spaces

For examples

(1) The function d on R defined by
$$d(x,y) = |x-y|$$
is a metric on R and it is known as usual metric on R

(2) Similarly the usual metric on C is defined by $d(z_1,z_2) = |z_1 - z_2|$

(3) Let X be a non empty set and d defined by
$$d(x,y) = \begin{cases} 0, & x = y \\ 1 & x \neq y \end{cases}$$
is a metric on X and is called discrete metric.

(iv) The metric d on X defined by
$$d(x,y) = \|x-y\|$$
is called the metric induced by the norm.

The most of the metric spaces of major importance in analysis are equipped with this type metrics i.e. these are normed linear spaces.

(v) The set $C[0,1]$ of all bounded continuous real functions defined on the closed unit interval $[0,1]$ is a metric space together with metric
$$d(x,y) = \|f-g\|$$
$$\text{with norm } \|f\| = \int_0^1 |f(x)|\,dx$$

Therefore $C[0,1]$ is metric space with metric
$$d(f,g) = \int_0^1 |f(x)-g(x)|\,dx$$

NOTE – $C[0,1]$ is also a metric space with another metric defined by
$$d(f,g) = \sup|f(x)-g(x)|$$

Subspace – Let (X,d) be a metric space. A nonempty subset Y of X is called subspace of (X,d) if (Y,d) is also metric space. e.g.
unit interval $[0,1]$ & set of rational numbers are subspace of the real line.

Pseudo Metric. - If in the above definition of the metric d, the ~~third~~ Second condition becomes

if $x = y \Rightarrow d(x, y) = 0$

then d is called a pseudo metric

e.g.

(i) $d(x, y) = |x^2 - y^2| \quad \forall\, x, y \in R$ is a pseudo metric

Distance from a point to a set

Let (X, d) be a metric space and let A be a subset of X. If $x_0 \in X$, then distance from x_0 to A is defined by

$$d(x_0, A) = \inf.\{d(x_0, x) : x \in A\}$$

$d(x_0, A) = +\infty \Leftrightarrow A$ is empty.

Diameter of a set

The diameter of a set A is defined by

$$d(A) = \sup\{d(x, y) : x, y \in A\}$$

The diameter of a bounded set is finite

The diameter of the empty set is infinite

Open and Closed spheres.

Let (X, d) be a metric space. If x_0 is a point in X and r is a positive real number

The open sphere $S_r(x_0)$ with centre x_0 and radius r is a subset of X defined by

$$S_r(x_0) = \{x \in X : d(x, x_0) < r\}$$

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The closed sphere $S_r[x_0]$ with centre x_0 and of radius r defined by

$$S_r[x_0] = \{x \in X : d(x_0, x) \leq r\}$$

An open sphere $S_\epsilon(x_0)$, where $\epsilon > 0$ is however small, is known as ϵ- nbd of x_0. Every superset containing $S_\epsilon(x_0)$ is also nbd of x_0. Every open sphere is nbd of its all points.

Interior point & Interior of a set:-

Let A be a subset of metric space (X, d). A point $x_0 \in A$ is called interior point of A if it is centre of some open sphere contained in A i.e. x_0 is an interior point of A if $\exists$ an open sphere $S_r(x_0) \subset A$.

The set of all interior points of A is called interior of A and is denoted by $Int(A)$ or A^o

$$\therefore \; Int(A) = \{x : x \in A \text{ and } S_r(x) \subset A \text{ for some } r > 0\}$$

OPEN SET —

A subset of a metric space (X, d) is open set if its every point is its interior point i.e. G is called an open set if for every $x \in G$ $\exists$ a positive real number r such that $S_r(x) \subset G$