Poisson Distribution

Let X be discrete r.v. that can take values 0, 1, 2, ---- such that probability function of X is given by $P(X=x) = \boxed{f(x) = \frac{\lambda^x e^{-\lambda}}{x!}}$, $x = 0, 1, 2, ----$

where $\lambda > 0$ is a constant. ↳ Poisson distribution

Relation b/w Poisson & Binomial distribution:

If in binomial distribution n is large and p (prob. of occurrence of an event) is close to zero (so that $q = 1-p$ is close to 1), then poisson distribution is used. For partical purposes if $n \geq 50$ ~~and~~ while $np \leq 5$ then ~~binomial~~ poisson distribution is used.

Ex A unfair coin is tossed 500 times. Find the probability of getting at least 2 heads given $P(\text{head}) = \frac{1}{100}$ ~~P(tails)~~

soln $n = 500$ $p = \frac{1}{100}$ $\therefore np = 5$ $n \geq 50$ & $np \leq 5$

$\therefore$ Poisson distribution can be used.

$\lambda = 5$ $\therefore P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$

$$= 1 - \left[\frac{5^0 e^{-5}}{0!} + \frac{5^1 e^{-5}}{1!}\right]$$

$$= 1 - 6e^{-5} \;//$$

Ex Establish validity of the Poisson approximation to the binomial distribution.

Soln Let X be binomially distributed, then

$$P(X=x) = \binom{n}{x} p^x q^{n-x} \quad \text{—} \circledast$$

$E(X) = np$ (for binomial distribution).

Let $\lambda = np \Rightarrow p = \frac{\lambda}{n}$. Then $\circledast$ becomes

$$P(X=x) = \binom{n}{x}\left(\frac{\lambda}{n}\right)^x \cancel{q}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)\cdots(n-x+1)}{x!\, n^x}\lambda^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$\longrightarrow \frac{\lambda^x}{x!}\cdot 1\cdot e^{-\lambda}\cdot 1 = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$\left(\because \left(1-\frac{\lambda}{n}\right)^{n-x} = \left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x} \to e^{-\lambda}\cdot 1 \text{ as } n\to\infty\right)$$

Hence $P(X=x) \to \frac{\lambda^x e^{-\lambda}}{x!}$ as $n\to\infty$.

which is Poisson distribution.

Ex Verify $P(X=x) = f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$ is a probability fn.

Soln (1) $f(x) \geq 0 \ \forall x = 0,1,2,\ldots$

(2) $\sum_{0}^{\infty} f(x) = \sum_{0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1$ //

Mean $= \mu = E(X) = \sum_{0} x f(x)$

$= \sum_{0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{\cancel{x!}\,(x-1)!} = \lambda \sum_{0}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!}$

$= \lambda e^{-\lambda} \sum_{0}^{\infty} \frac{\lambda^y}{y!}$ where $y = (x-1)$

$= \lambda e^{-\lambda} . e^{\lambda} = \lambda$ //

Variance: $Var(X) = E(X^2) - [E(X)]^2$.

Consider $E(X(X-1)) = \sum_{\cancel{0}}^{\infty} \frac{x(x-1)\lambda^x e^{-\lambda}}{\cancel{x!}(x-2)!}$

$= e^{-\lambda} \lambda^2 \sum_{2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} = e^{-\lambda} \lambda^2 . e^{\lambda}$

$= \lambda^2$.

Now, $E(X^2) = E(X(X-1) + X] = E[X(X-1)] + E(X)$.

$= \lambda^2 + \lambda$.

$\therefore Var(X) = E(X^2) - [E(X)]^2 = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \lambda$ //.

standard deviation: $\sigma = \sqrt{\lambda}$.

Moment generating fn:

$$M(t) = E(e^{tX}) = \sum_0^\infty e^{tx} \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_0^\infty \frac{(\lambda e^t)^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_0^\infty \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)} \quad //$$

Example: If 3% of the electric bulbs manufactured by a company is defective, find probability that in a sample of 100 bulbs (a) 0 (b) 3 will be defective. Also find third moment about mean.

Soln $n = 100$ $p = \frac{3}{100}$ $np = 3 \leq 5$ $\therefore$ Poisson distribution

$\lambda = 5$. $X \to$ no. of def. bulbs

(1) $P(X=0) = \frac{5^0 e^{-5}}{0!} = e^{-5}$.

(2) $P(X=3) = \frac{5^3 e^{-5}}{3!} = \frac{125 e^{-5}}{6}$.

(3) $\mu = \sigma^2 = \lambda$. $\mu_2 = \sigma^2 = \lambda$.

$\Rightarrow \mu = \mu_1 = \mu_1' = \lambda$

$$\mu_3' = \frac{d^3}{dt^3} M(t)\Big|_{t=0} = \frac{d^3}{dt^3}\left[e^{5(e^t - 1)}\right]\Big|_{t=0} = 45.$$

$\mu_2' = 5$ $\left(= \frac{d^2}{dt^2} M(t)\Big|_{t=0}\right)$

$$\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 3\mu_1' \mu_1'^2.$$

$$= 45 - 3 \cdot \cancel{5}\,5 + 3 \cdot \cancel{5 \cdot 5} = 45 \quad //$$