Some theorems on Random Variables: If X and Y are random variables and c_1 and c_2 are two constants then

1. $c_1X + c_2Y$ is also a random variable
   [$\therefore$ $X+Y$ & $X-Y$ are random variables.
2. XY is also a random variable
3. $\max[X, Y]$ and $\min[X, Y]$ are also random variables

Properties of distribution function:

Prop 1: If $F(x)$ is distribution function of random variable X and if $a < b$ then

$$P(a < X \leq b) = F(b) - F(a)$$

Pf: let $A: a < X \leq b$ & $B: X \leq a$.

Then $A \cap B = \phi$ & $A \cup B: X \leq b$

Hence, $P(X \leq b) = P(X \leq a) + P(a < X \leq b)$ (addition thm of prob)

$\Rightarrow P(a < X \leq b) = P(X \leq b) - P(X \leq a)$
$= F(b) - F(a)$

Prop 2. $P(a \leq X \leq b) = F(b) - F(a) + P(X = a)$

Pf: $P(a \leq X \leq b) = P(X = a) + P(a < X \leq b)$
$= P(X = a) + F(b) - F(a)$ (from Prop 1)

Prop 3: $P(a < X < b) = F(b) - F(a) - P(X = b)$

Pf: $P(a < X < b) = P(a < X \leq b) - P(X = b)$ (from prop 1)
$= F(b) - F(a) - P(X = b)$

Prop 4: $P(a \le X < b) = F(b) - F(a) - P(X=b) + P(X=a)$

Pf: $P(a \le X < b) = P(a < X < b) + P(X=a)$

$= F(b) - F(a) - P(X=b) + P(X=a)$ (from Prop 3)

Prop 5: If $F(x)$ is distribution function for random variable X, then $F(x_1) \le F(x_2)$ if $x_1 < x_2$

Pf: $F(x_2) - F(x_1) = P(x_1 < X \le x_2) \geq 0$ $(\because P(A) \geq 0)$

$\Rightarrow F(x_2) \geq F(x_1)$ for $x_1 < x_2$

Ex: A discrete random variable takes the values $-3, -2, -1, 0, 1, 2, 3$ such that $P(X=-3) = P(X=-2) = P(X=-1)$

$P(X=1) = P(X=2) = P(X=3)$

and $P(X=0) = P(X>0) = P(X<0)$

Obtain (1) pmf & distribution function of X

(2) " " " " $Y = 2X^2 + 3X + 4$

Soln (1)

X	-3	-2	-1	0	1	2	3
$P(x)$	$1/9$	$1/9$	$1/9$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

let $P(X=-3) = P(X=-2) = P(X=-1) = \alpha$

$P(X=1) = P(X=2) = P(X=3) = \beta$ & $P(X=0) = \gamma$

Then
$$\left.\begin{array}{l} 3\alpha + 3\beta + \gamma = 1 \\ 3\alpha = \gamma \quad (\because P(X=0) = P(X>0)) \\ 3\beta = \gamma \quad (\because P(X=0) = P(X<0)) \end{array}\right] \Rightarrow \alpha = \beta = \frac{1}{9}$$

(2) $Y = 2X^2 + 3X + 4$

$\therefore Y = 13, 6, 3, 4, 9, 18, 31$

$\therefore$

y	13	6	3	4	9	18	31
$g(y)$	1/9	1/9	1/9	1/3	1/9	$\frac{1}{9}$	$\frac{1}{9}$

$y = 13 = 2x^2 + 3x + 4 \Rightarrow x = -3.$

$\therefore g(13) = f(-3) = 1/9$ $\qquad$ $y = 6 \Rightarrow x = -2$

$\therefore g(6) = f(-2) = \frac{1}{9}$

and so on . .

Ex The following is distribution function of discrete random variable X:

x	-3	-1	0	1	2	3	5	8
$F(x)$	0.10	0.30	0.45	0.5	0.75	0.90	0.95	1.00

Find (1) Probability mass function of X

(2) $P(X \text{ even})$ & $P(X \text{ is odd})$

(3) $P(X = -3 \mid X < 0)$ & $P(X \geq 3 \mid X > 0)$

Soln (1) $f(x_i) = F(x_i) - F(x_{i-1})$

x	-3	-1	0	1	2	3	5	8
$f(x)$	0.10	0.20	0.15	0.05	0.25	0.15	0.05	0.05

(2) $P(X \text{ even}) = f(2) + f(8) = 0.30$

$P(X \text{ odd}) = f(-3) + f(-1) + f(1) + f(3) + f(5) = 0.55$

(3) $P(X = -3 \mid X < 0) = \dfrac{P(X = -3)}{P(X < 0)} = \dfrac{f(-3)}{f(-3) + f(-1)} = \dfrac{1}{3}$

$P(X \geq 3 \mid X > 0) = \dfrac{P(X \geq 3)}{P(X > 0)} = \dfrac{f(3) + f(5) + f(8)}{f(1) + f(2) + f(3) + f(5) + f(8)} = \dfrac{5}{11}$