Evaluation of Real Integrals by Method of Contour Integration

A large number of real integrals whose evaluation by usual methods is some times very tedious can be easily evaluated by using cauchy's residue theorem

In order to evaluate real integrals by using residue theorem we require a closed curve C, called contour, so as per our syllabus and type C we shall discuss evaluation of two types of real integrals

Type-I - Evaluation of real integral of form $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

To evaluate such integrals we take C as unit circle $|z|=1$ and put $z = e^{i\theta}$ to convert the integrand in term of z

Ex.1 Using calculus of residue. Show that

$$\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\, d\theta = \frac{\pi a^2}{1-a^2} \quad (a^2<1)$$

Consider

$$I = \int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}$$

$$= R.P. \int_0^{2\pi} \frac{e^{2i\theta}}{1-2a\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)+a^2}\, d\theta$$

$$I = R.P. \int_C \frac{z^2}{1-a(z+\frac{1}{z})+a^2} \frac{dz}{iz}$$

where C is $|z|=1$

$\Rightarrow z = e^{i\theta}$

$dz = ie^{i\theta} d\theta$

i.e. $d\theta = \frac{dz}{iz}$

$$= R.P. \frac{1}{i} \int$$

$$I = R.P. \frac{1}{i} \int_C \frac{z^2}{(1+a^2)z - az^2 - a}\, dz$$

Scanned with CamScanner

(4)

$$I = R.P.\ i\int_C \frac{z^2}{az^2-(1+a^2)z+a}\,dz \qquad (1)$$

let $f(z) = \dfrac{z^2}{az^2-(1+a^2)z+a}$

for poles of $f(z)$, $az^2-(1+a^2)z+a=0$

$\Rightarrow z = a, \ \& \ z = 1/a$

$\because a^2<1$ i.e. $|a|<1 \Rightarrow$ only one pole $z=a$ lies within C and we have.

$$f(z) = \frac{z^2}{a(z-a)(z-1/a)}$$

$$Res(z=a) = \lim_{z\to a}(z-a)f(z) = \frac{-a^2}{(1-a^2)}$$

and then by Cauchy's residue theorem

$$\int_C \frac{z^2}{az^2-(1+a^2)z+a}\,dz = 2\pi i\ Res(z=a) = -\frac{2\pi i a^2}{(1-a^2)}$$

putting this value in (1), we have.

$$I = R.P.\ i \times \left(\frac{-2\pi i a^2}{1-a^2}\right) = R.P.\left(\frac{2\pi a^2}{1-a^2}\right)$$

$$I = \frac{2\pi a^2}{(1-a^2)}$$ Hence proved.

Type-II – Integral of type $\int_{-\infty}^{\infty} f(x)\,dx$

where $f(z)$ has poles in upper half z-plane and having no poles on the real axis. We evaluate the above integral by considering them along a closed contour C consisting of

(i) Semi circle $|z|=R$ in the upper half plane

(ii) real axis from $-R$ to R.

Important Results :- To evaluate the above integrals we use one of the following results (theorems)

Theorem-1 If C is an arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z|=R$ and $\lim_{z\to\infty} z f(z) = K$ then

$$\lim_{R\to\infty} \int_C f(z)\,dz = i(\theta_2-\theta_1)K$$

Theorem-2 **Jordan's Lemma** – If $f(z) \to 0$ as $|z|\to\infty$, uniformly for $0 \le \arg(z) \le \pi$ and $f(z)$ is meromorphic in the upper half plane then

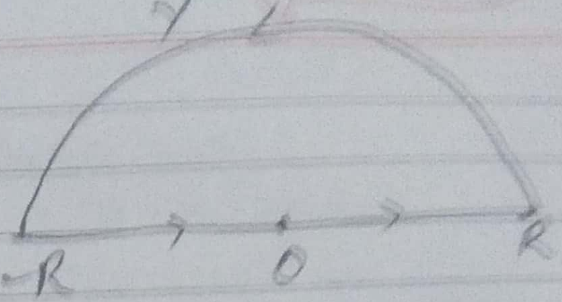
$$\lim_{R\to\infty} \int_\gamma e^{imz} f(z)\,dz = 0 \quad (m>0)$$

where γ denotes semi circle $|z|=R$, $Im(z)>0$

(Ex. 2) – By Using Method of contour integration show that $\displaystyle\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}$

Soln. Consider $I = \displaystyle\int_C \frac{1}{z^4+a^4}\,dz$

where C is closed contour consisting of of upper semi circle γ of the circle $|z|=R$ and real axis from $-R$ to R.



Here

$$f(z) = \frac{1}{z^4+a^4}$$

for poles: $z^4 + a^4 = 0$

$$z = a(-1)^{1/4}$$

$$\Rightarrow z = ae^{i\pi/4},\ ae^{i3\pi/4},\ ae^{i5\pi/4},\ ae^{i7\pi/4}$$

on which first two poles lie in the upper half i.e. within C.

$$\text{Res}(z = ae^{i\pi/4}) = \lim_{z \to ae^{i\pi/4}} (z - ae^{i\pi/4}) \cdot \frac{1}{z^4+a^4}$$

$$= \lim_{z \to ae^{i\pi/4}} \frac{1}{4z^3} \quad \text{by L-Hospital Rule}$$

$$= \frac{e^{-i3\pi/4}}{4a^3}$$

Similarly $\text{Res}(z = e^{i3\pi/4}) = \frac{e^{-i9\pi/4}}{4a^3} = \frac{e^{-i\pi/4}}{4a^3}$

Now by Cauchy's Residue theorem

$$\int_C \frac{1}{z^4+a^4}\,dz = 2\pi i\,[\text{Res}(z=ae^{i\pi/4}) + \text{Res}(z=e^{i3\pi/4})]$$

$$= 2\pi i \left[\frac{e^{-i3\pi/4}}{4a^3} + \frac{e^{-i\pi/4}}{4a^3}\right]$$

$$= \frac{-2\pi i}{4a^3}\, i\sqrt{2} = \frac{\pi\sqrt{2}}{2a^3} \quad \text{—— (1)}$$

Further by structure of C, we have

$$\int_C f(z)\,dz = \int_\gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx$$

$$\Rightarrow \int_C \frac{1}{z^4+a^4}\,dz = \int_\gamma \frac{1}{z^4+a^4}\,dz + \int_{-R}^{R} \frac{1}{x^4+a^4}\,dx$$

—— (2)

from ① & ② we have. (7)

$$\int_\gamma \frac{1}{z^4+a^4}\,dz + \int_{-R}^{R} \frac{1}{x^4+a^4}\,dx = \frac{\pi\sqrt{2}}{2a^3}$$

making $R \to \infty$

$$\lim_{R\to\infty} \int_\gamma \frac{1}{z^4+a^4}\,dz + \int_{-\infty}^{\infty} \frac{1}{x^4+a^4}\,dx = \frac{\pi\sqrt{2}}{2a^3} \quad \text{--- ③}$$

since $\lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z}{z^4+a^4} = 0$

then by theorem ①.

$$\lim_{R\to\infty} \int_\gamma \frac{1}{z^4+a^4}\,dz = i(\pi-0)\cdot 0$$
$$= 0$$

using it in ③ we get

$$\int_{-\infty}^{\infty} \frac{1}{x^4+a^4}\,dx = \frac{\pi\sqrt{2}}{2a^3}$$

and $$\int_{0}^{\infty} \frac{1}{x^4+a^4}\,dx = \frac{\pi\sqrt{2}}{4a^3}.$$

Scanned with CamScanner