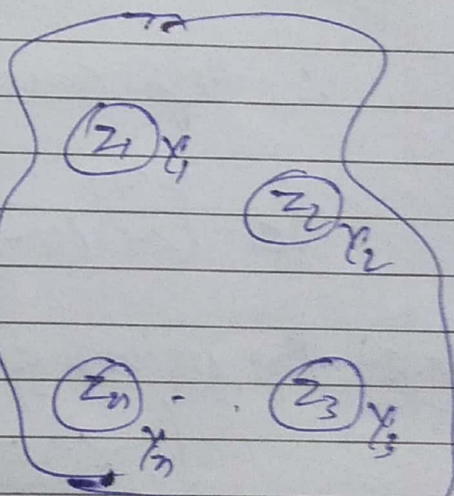
Cauchy Residue Theorem

If $f(z)$ is analytic, except at a finite number of poles within a closed contour C ~~above its residue~~ and continuous on the boundary C then

Proof:- Let $f(z)$ be analytic function except finite number of poles $z_1, z_2 \ldots z_n$ within C, Draw the circles $\gamma_1, \gamma_2 \ldots \gamma_n$ with centres at $z_1, z_2 \ldots z_n$ respectively and each of radius ρ which is so small that all these circles lie entirely within C and do not overlap. Then $f(z)$ is analytic in the region enclosed by C and these circles. Hence by Cauchy's theorem for multiply connected region, we have



$$\int_C f(z) - \sum_{r=1}^{n} \int_{\gamma_r} f(z)\,dz = 0$$

or
$$\int_C f(z) = \sum_{r=1}^{n} \int_{\gamma_r} f(z)\,dz$$

$$\int_C f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \cdots + \int_{\gamma_n} f(z)\,dz \quad (1)$$

But by def. of residue, we have.

$$\text{Res}(z = z_0) = \frac{1}{2\pi i} \int_{\gamma} f(z)\,dz$$

or
$$\int_{\gamma} f(z)\,dz = 2\pi i \,\text{Res}(z = z_0) \quad (2)$$

In view of (2), eq^n (1) can be written as

$$\int_C f(z) = 2\pi i \left[\text{Res}(z=z_1) + \text{Res}(z=z_2) + \cdots + \text{Res}(z=z_n)\right]$$

$\int_C f(z) = 2\pi i$ sum of residue at poles lying within C.

Hence proved.

e.g. Using Cauchy's residue theorem evaluate $\int_C \frac{z^2+1}{(z-1)(z-2)}\, dz$

where C is $|z| = 3/2$

Soln:

let $f(z) = \frac{z^2+1}{(z-1)(z-2)}$

The poles of $f(z)$ are $z=1$ & $z=2$ but only one simple pole $z=1$ lies within C

$$Res(z=1) = \lim_{z \to 1} (z-1) f(z)$$

$$= \lim_{z \to 1} \frac{z^2+1}{(z-2)} = -2$$

$\therefore$ By Cauchy's residue theorem

$$\int_C \frac{z^2+1}{(z-1)(z-2)}\, dz = 2\pi i\, Res(z=1)$$

$$= -4\pi i$$

Ex. 2 By using calculus of residue evaluate

$$\int_C \frac{e^z}{(z-3)^2(z-2)}\, dz$$

where C is $|z| = 4$

Soln: $f(z) = \frac{e^z}{(z-3)^2(z-2)}$

$f(z)$ has a simple pole $z=2$ and $z=3$ of order 2 and both lie within C then by Cauchy's residue theorem

$$\int_C \frac{e^z}{(z-3)^2(z-2)}\, dz = 2\pi i \left[Res(z=2) + Res(z=3) \right]$$

$$= 2\pi i \left[e^2 + 0 \right] = 2\pi i e^2$$

Scanned with CamScanner