# Residue at pole.

Let $z = a$ is an isolated singularity of $f(z)$ then by Laurent's expansion of $f(z)$, we have.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

The coefficient of $\frac{1}{(z-a)}$ i.e, $b_1$ is called residue of $f(z)$ at $z = a$ is given by.

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\,dz$$

and is denoted by $\text{Res}_{z=a} f(z)$ or $\text{Res}(z=a)$

OR. If $z = a$ is the only singularity of an analytic function $f(z)$ inside a closed curve $C$, and if $\frac{1}{2\pi i} \int f(z)\,dz$ has a definite value then that value is called the residue of $f(z)$ at $z = a$

## Calculation of Residues.

(1) If $z = a$ is a simple pole of $f(z)$ then

$$\text{Res}(z=a) = \lim_{z \to a} (z-a) f(z)$$

(2) If $z = a$ is a simple pole of $f(z)$ and $f(z) = \frac{\phi(z)}{\psi(z)}$ then

$$\text{Res}(z=a) = \frac{\phi(a)}{\psi'(a)}$$

(3) If $z = a$ is a pole of order $m$, then

$$Res(z=a) = \lim_{z \to a} \left[ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z) \right]$$

Examples :-

Ex.1 Find the residue of $f(z) = \frac{z^2+1}{z^2-3z+2}$ at the poles

Soln.

$$f(z) = \frac{1+z^2}{z^2-3z+2} = \frac{1+z^2}{(z-1)(z-2)}$$

Here $z=1$ & $z=2$ are simple poles of $f(z)$

$$\therefore Res(z=1) = \lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} \frac{1+z^2}{(z-2)}$$

$$= -2$$

and $Res(z=2) = \lim_{z \to 2} (z-2) f(z) = \lim_{z \to 2} \frac{z^2+1}{(z-1)} = 5$

Ex.2 Find the residue of $\frac{1}{(z^2+a^2)^2}$ at $z = ia$

$$f(z) = \frac{1}{(z^2+a^2)^2} = \frac{1}{(z-ia)^2 (z+ia)^2}$$

Here $z = ia$ is a pole of order 2

$$\therefore Res(z=ia) = \lim_{z \to ia} \left[ \frac{1}{1!} \frac{d}{dz} (z-ia)^2 f(z) \right]$$

$$= \lim_{z \to ia} \left[ \frac{d}{dz} \frac{1}{(z+ia)^2} \right]$$

$$= \lim_{z \to ia} \frac{-2}{(z+ia)^3} = \frac{-2}{(2ia)^3} = \frac{1}{4ia^3}$$

REMARK – since residue of $f(z)$ at $z = a$ is the coeff. of $\frac{1}{z-a}$ so some time it is convenient to find Res $(z = a)$ of $f(z)$ as the coeff. of $\frac{1}{t}$ in the expansion of $f(a+t)$

e.g. Find the residue of $\frac{z^2}{(z-1)^4(z-2)}$ at $z = 1$

Soln. let $f(z) = \frac{z^2}{(z-1)^4(z-2)}$ and $z=1$ is pole of order 4

$\text{Res}(z=1) = $ coeff. of $\frac{1}{t}$ in expansion of $f(1+t)$

$= $ coeff. of $\frac{1}{t}$ in $\frac{(1+t)^2}{t^4(t-1)}$

$= $ coeff. of $\frac{1}{t}$ in $\left\{\frac{-1}{t^4}(1+t^2+2t)(1-t)^{-1}\right\}$

$= $ coeff. of $\frac{1}{t}$ in $\left\{\frac{-1}{t^4}(1+t^2+2t)(1+t+t^2 + \cdots\right.$

$= $ coeff. of $\frac{1}{t}$ in $\left\{\frac{-1}{t^4}\left\{1+2t^2+2t^3+ \cdots\right\}\right\}$

$= -$ coeff. of $\frac{1}{t}$ in $\left\{\frac{1}{t^4} + 2/t^2 + 2/t + \cdots\right\}$

$= -2$ Ans.

Residue at infinity :-

By def. $\text{Res}(z=\infty) = -\frac{1}{2\pi i}\int_C f(z)\,dz$

or (1) $\text{Res}(z=\infty) = \lim_{z\to\infty} -z f(z)$

(provided it has a definite value)

(2) $\text{Res}(z=\infty) = -$ coeff. of $\frac{1}{z}$ in the expansion of $f(z)$.

(Ex.3) Find the residue of $\dfrac{z^3}{z^2-1}$ at $z=\infty$

Let $f(z) = \dfrac{z^3}{z^2-1}$

$$\text{Res}(z=\infty) = \lim_{z\to\infty} -z f(z)$$

$$= \lim_{z\to\infty} \frac{-z^4}{z^2-1}$$

does not exist

Now we have

$$f(z) = \frac{z^3}{z^2(1-1/z^2)} = z(1-1/z^2)^{-1}$$

$$= z\left\{1 + \frac{1}{z^2} + \frac{1}{z^4} + \cdots\right\}$$

$$= z + \frac{1}{z} + \frac{1}{z^3} + \cdots$$

and $\text{Res}(z=\infty) = -$ coeff. of $\frac{1}{z}$ in the expansion of $f(z)$

$$= -1$$