Singularities -

① **Poles** - If $f(z)$ has form

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + \frac{a_{-1}}{z-a} + \frac{a_{-1}}{(z-a)^2}$$

in which the principle part has only a finite number of terms given by,

$$\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \dots + \frac{a_{-n}}{(z-a)^n}$$

where $a_n \neq 0$, $z = a \Rightarrow$ is called a pole of order n.

If $n = 1$ simple pole

If $f(z)$ has a pole at $z = a$, then $\lim_{z \to a} f(z) = \infty$

② **Branch Point** - A point $z = z_0$ is called a branch point of the multiple-valued function $f(z)$ if the branches of $f(z)$ are interchanged when z describes a closed path about z_0. A Branch point is a non-isolated singularity.

Ex. $f(z) = z^{1/2}$ which has the value 1 for $z = 1$, has a Taylor series of the form $a_0 + a_1(z-1) + a_2(z-1)^2 + \dots$ with radius of convergence $R = 1$.

③ **Removable Singularities** - If a single-valued function $f(z)$ is not defined at $z = a$ but $\lim_{z \to a} f(z)$ exists, then $z = a \rightarrow$ is called removable singularity.

Ex- $f(z) = \frac{\sin z}{z}$ then $z = 0$ is a removable singularity.

since $f(0)$ is not defined but $\lim_{z \to a} \frac{\sin z}{z} = 1$

Essential singularities - If $f(z)$ is single-valued then any singularity which is not a pole or removable singularity is called an essential singularity. If $z=a$ is an essential singularity of $f(z)$, the principle part of the Laurent expansion has infinitely many terms.

Ex- $$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$ $z=0$ is an essential singularity.

Residues - Let $f(z)$ be single-valued and analytic inside and on a circle C except at the point $z=a$ chosen as the center of C. Laurent series about $z=a$ is given by: we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots \quad -①$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz, \quad n = 0, \pm 1, \pm 2 \quad -②$$

If $n=-1$ $$\oint_C f(z)\, dz = 2\pi i a_{-1} \quad -③$$

$$\oint_C \frac{dz}{(z-a)^p} = \begin{cases} 2\pi i & p=1 \\ 0 & p=\text{integer} \end{cases}$$