DENSE And NON-DENSE SETS

Let (X, T) be topological space and A, B be subsets of X then

(i) A is said to be dense in B iff $\overline{B} \subset \overline{A}$

(ii) A is said to be dense in X or everywhere dense iff $\overline{A} = X$

It follows that A is everywhere dense iff every point of X is an adherent point of A.

(iii) A is said to be nowhere dense or non-dense iff $(\overline{A})^0 = \phi$ i.e. interior of closure of A is empty.

(iv) A is said to be dense in itself iff $A \subset D(A)$

A set $A \subset X$ is perfect iff A is dense in itself and A is closed i.e. $A = D(A)$

Separable spaces: A topological space X is said to be separable iff X contains a countable dense (everywhere dense) subset i.e. iff $\exists$ a countable subset A s.t. $\overline{A} = X$.

e.g. The usual topological space (R, T) is separable since the Q is the countable dense subset of R.

Hausdorff Space :- A topological space (X, T) is said to be a Hausdorff space (or a separated space or a T_2-space) iff for every pair of distinct points x, y of X, there exist disjoint nbds of x and y.

If (X, T) be a Hausdorff space then T is called Hausdorff topology for X.

(i) Every discrete space is Hausdorff.

(ii) $(\mathbb{R}, U)$ and $(\mathbb{R}, S)$ be Hausdorff space.

(iii) Consider $X = \{a, b, c\}$ and let

$$T_1 = \{\phi, \{a\}, \{b, c\}, X\}$$

then (X, T_1) is not a Hausdorff space since $\exists$ no disjoint nbds of b & c.

SUBSPACE– let (X, T) be a topological space and let Y is a subset of X then the collection

$$T_Y = \{G \cap Y : G \in T\}$$ is a topology on Y

and it is called the relative topology to Y or the topology induced by T.

and then the topological space (Y, T_Y) is called a subspace of (X, T)

Ex. Consider the topology

$$T = \{\{1\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4, 5\}, X, \phi\}$$

on $X = \{1, 2, 3, 4, 5\}$

Find the relative topology for $Y = \{1, 4, 5\}$

Soln. we have

$Y \cap \{1\} = \{1\}$, $\quad Y \cap \{3, 4\} = \{4\}$.

$Y \cap \{1, 3, 4\} = \{1, 4\}$ $\quad Y \cap \{2, 3, 4, 5\} = \{4, 5\}$

$Y \cap X = Y$ and $Y \cap \phi = \phi$

Hence the relative topology to Y is

$$T_Y = \{\{1\}, \{4\}, \{1, 4\}, \{4, 5\}, Y, \phi\}$$

and (Y, T_Y) is a subspace of (X, T)