Symmetries of a Square :-

Let us consider the scenario of removing a square from a plane. If we move it in some way and put the square back into the space it originally occupied.

Our goal is to discribe the possible relationships between the starting position of the square and its final position in terms of motions.

For instance, we consider a 90° rotation and a 450° rotation as equal, since they have the same effect on every point.

We can think square as being transparent (glass say) with corners marked on one side with the colors blue, white, pink and green so that motions effect can be easily distinguished.

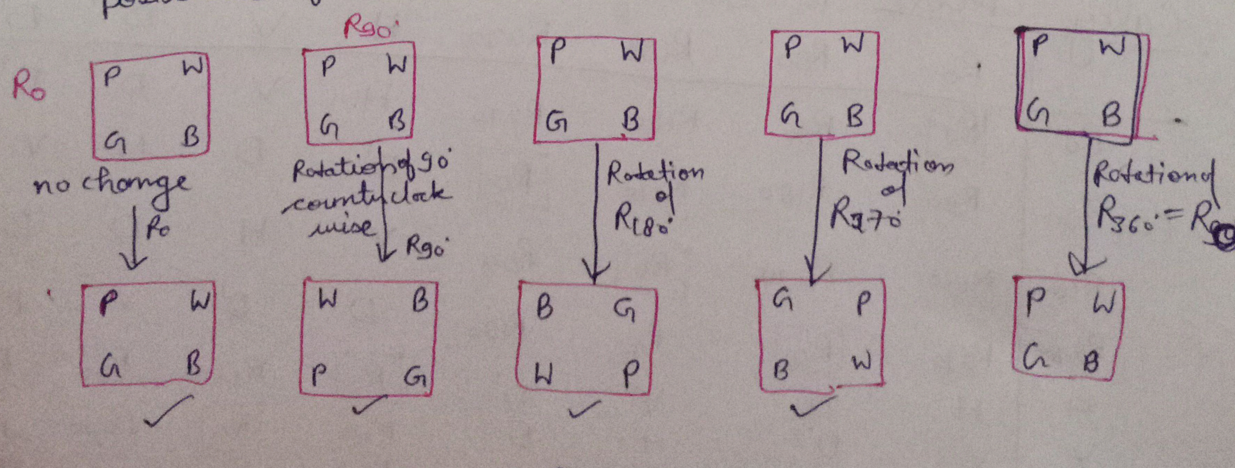
Observation :- Final position of the square is completely determined by the location and orientation

location → face up

orientation → face down

clearly, total 4 locations } for a given corner
total 2 orientations

so there are exactly eight distinct final positions for the corner.



H: Rotation of 180° about a ~~vertical~~ axis horizontal

$$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xrightarrow{H} \begin{bmatrix} G & B \\ P & W \end{bmatrix}$$

V: Rotation of 180° about a vertical axis

$$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xleftrightarrow{V} \begin{bmatrix} W & P \\ B & G \end{bmatrix}$$

D: Rotation of 180° about the main diagonal.

$$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xleftrightarrow{D} \begin{bmatrix} P & G \\ W & B \end{bmatrix}$$

D': Rotation of 180° about the other diagonal.

$$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xrightarrow{D'} \begin{bmatrix} B & W \\ G & P \end{bmatrix}$$

Thus these eight motions $\underline{R_0, R_{90°}, R_{180°}, R_{270°}}$, ~~R360~~, $\underline{H, V, D, D'}$ together with the operation composition, form a Mathematical system called the dihedral group of order 8 say D_4

Observation:

$$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xrightarrow{R_{90°}} \begin{bmatrix} W & B \\ P & G \end{bmatrix} \xrightarrow{H} \begin{bmatrix} P & G \\ W & B \end{bmatrix}$$

$$HR_{90} = D$$

Caley Table

R_0 → identity

$HD \neq DH$

	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0	$\boxed{R_0}$	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	$\boxed{R_0}$	D'	D	H	V
R_{180}	R_{180}	R_{270}	$\boxed{R_0}$	R_{90}	V	H	D'	D
R_{270}	R_{270}	$\boxed{R_0}$	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	$\boxed{R_0}$	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	$\boxed{R_0}$	R_{270}	R_{90}
D	D	V	D'	H	R_{270}	R_{90}	$\boxed{R_0}$	R_{180}
D'	D'	H	D	V	R_{90}	R_{270}	R_{180}	$\boxed{R_0}$