" A series converges to a number S iff the sequence of remainders converges to zero."

Taylor (Series) Theorem :- If a function $f(z)$ is analytic within a circle C with the centre $z = z_0$ and radius R then at every point z inside C

$$f(z) = f(z_0) + (z-z_0)\frac{f'(z_0)}{1!} + (z-z_0)^2\frac{f''(z_0)}{2!} + \cdots + (z-z_0)^n\frac{f^n(z_0)}{n!} + \cdots$$

that is, the power series converges to $f(z)$ when $|z-z_0| < R$. ~~Let z be any point within C s.t $|z-z_0| < r$~~

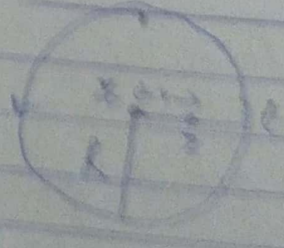
Proof

Let $f(z)$ be analytic within a circle C whose eq^n is $|t-z_0| = R$. Let z be any point within C s.t $|z-z_0| = r < R$

By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t-z}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-z_0)-(z-z_0)}$$



$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-z_0)}\left[1 - \frac{z-z_0}{t-z_0}\right]^{-1} dt$$

$$f(z) = \frac{1}{2\pi i}\int \frac{f(t)}{(t-z_0)}\left[1 + \frac{z-z_0}{t-z_0} + \left(\frac{z-z_0}{t-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{t-z_0}\right)^3\right.$$
$$\left. + \left(\frac{z-z_0}{t-z_0}\right)^{n+1}\left(1 - \frac{z-z_0}{t-z_0}\right)^{-1}\right] dt$$

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[For $\frac{1}{1-b} = (1-b)^{-1} = 1+b+b^2+\cdots+b^n+\frac{b^{n+1}}{1-b}$]

using the formula

$$\frac{f^n(a)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$ we set

$$f(z) = f(z_0) + (z-z_0)\frac{f'(z_0)}{1!} + (z-z_0)^2\frac{f''(z_0)}{2!} + \cdots + (z-z_0)^n\frac{f^n(z_0)}{n!} + R_n(z) \quad \text{--- (1)}$$

where $R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i}\int \frac{f(t)\,dt}{(t-z)(t-z_0)^{n+1}}$

Since $f(z)$ is analytic everywhere $|f(t)| < M$

$$|R_n(z)| \le \frac{|z-z_0|^{n+1}}{2\pi}\int_C \frac{|f(t)||dt|}{(|t-z_0|-|z-z_0|)(t-z_0)^{n+1}}$$

$$\le \frac{M}{2\pi}\left(\frac{r}{R}\right)^{n+1}\frac{1}{(R-r)}\cdot 2\pi R$$

or $|R_n(z)| \le M\left(\frac{r}{R}\right)^{n+1}\frac{1}{1-r/R} \to 0$ as $n\to\infty$

$\because \frac{r}{R} < 1.$

$\therefore \lim_{n\to\infty} R_n(z) = 0$

~~Making~~ Thus for each point z interior to C, the limit as $n\to\infty$

$$f(z) = f(z_0) + (z-z_0)\frac{f'(z_0)}{1!} + (z-z_0)^2\frac{f''(z_0)}{2!} + \cdots + (z-z_0)^n\frac{f^n(z_0)}{n!} + \cdots$$

If $f(z)$ is regular in the finite plane

then Taylor's series about $Z=0$ for $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n Z^n$$

And Taylor's series for $f(1/z)$ about $Z=0$

will be $f(1/z) = \sum_{n=0}^{\infty} a_n Z^{-n}$

Deduction - (1) If we write $z_0 = 0$, we get

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \dots + \frac{z^n}{n!} f^n(0)$$

This is known as Maclaurin's series

(2.) The domain of convergence of Taylor series is given by $|z - z_0| < R$ where the radius R of convergence is the distance from z_0 to the nearest singularity of the function $f(z)$. On the circle $|z - z_0| = R$ the series may or may not converge.

Theorem (Laurent's Theorem) -

Suppose a function $f(z)$ is analytic in the closed ring bounded by two concentric circles C and C' of the centre z_0 and radii R and R' $(R' < R)$. If z is any point of the annulus then at each point z in the annular domain $f(z)$ is represented by the expression

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(t)\, dz}{(t - z_0)^{n+1}}$$

and

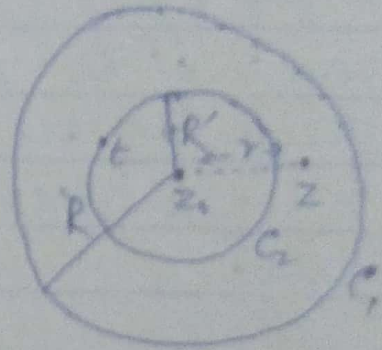
$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(t)\, dz}{(t - z_0)^{-n+1}}$$

The series is called Laurent series.

Proof:- Let $f(z)$ be analytic in the closed ring bounded by two concentric circles C_1 and C_2 of centre z_0 and radii R and R' ($R' < R$). Then if z is any point within the ring space, then

$$R' < |z - z_0| = r < R$$

Here we shall make use of following facts:



(i) $\frac{1}{1-b} = (1-b)^{-1}$

$$= 1 + b + b^2 + \cdots + b^n + \frac{b^{n+1}}{1-b}$$

(ii) $\left[1 - \frac{t - z_0}{z - z_0}\right]^{-1} = \frac{1}{1 - (t - z_0)/(z - z_0)} = \frac{z - z_0}{z - t}$

(iii) $\lim_{n \to \infty} \left(\frac{r}{R}\right)^n = 0 = \lim_{n \to \infty} \left(\frac{R'}{r}\right)^n$ as $\frac{r}{R} < 1$, $\frac{R'}{r} < 1$

(iv) $\int |dt| = 2\pi \cdot$ radius of circle $=$ circumference.

By [illegible] by Cauchy's integral formula

$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{t-z}\,dt - \frac{1}{2\pi i}\int_{C_2} \frac{f(t)}{(t-z)}\,dt$

$= \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{t-z}\,dt + \frac{1}{2\pi i}\int \frac{f(t)}{(z-t)}\,dt$

$= \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{(t-z_0)-(z-z_0)}\,dt + \frac{1}{2\pi i}\int_{C_2} \frac{f(t)\,dt}{(z-z_0)-(t-z_0)}$

$= \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{(t-z_0)}\left\{1-\frac{z-z_0}{t-z_0}\right\}^{-1} dt + \frac{1}{2\pi i}\int_{C_2} \frac{f(t)}{(z-z_0)}\left\{1-\frac{t-z_0}{z-z_0}\right\}^{-1} dt$

$= \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{(t-z_0)}\left[1+\left(\frac{z-z_0}{t-z_0}\right)+\left(\frac{z-z_0}{t-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{t-z_0}\right)^n + \left(\frac{z-z_0}{t-z_0}\right)^{n+1}\left\{1-\left(\frac{z-z_0}{t-z_0}\right)\right\}^{-1}\right] dt$

$+ \frac{1}{2\pi i}\int_{C_2} \frac{f(t)}{(z-z_0)}\left[1+\left(\frac{t-z_0}{z-z_0}\right)+\left(\frac{t-z_0}{z-z_0}\right)^2+\cdots+\left(\frac{t-z_0}{z-z_0}\right)^n + \left(\frac{t-z_0}{z-z_0}\right)^{n+1}\left\{1-\left(\frac{t-z_0}{z-z_0}\right)\right\}^{-1}\right] dt$

Taking $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{(t-z_0)^{n+1}}\,dt$

$b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(t)}{(t-z_0)^{-n+1}}\,dt$

$f(z) = \left[a_0 + (z-z_0)a_1 + (z-z_0)^2 a_2 + \cdots + a_n(z-z_0)^n + R_n(z)\right]$

$+ \left[\frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + R_n'(z)\right]$

where $R_n(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{t-z}\left(\frac{z-z_0}{t-z_0}\right)^{n+1} dt$

$R_n'(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(t)}{(z-t)}\left(\frac{t-z_0}{z-z_0}\right)^{n+1} dt$

Let $M = \max|f(t)|$ on C_1 and $M' = \max|f(t)|$ on C_2

$$|\rho_n(z)| \le \frac{1}{2\pi}\int_{C_1} |f(t)| \left|\frac{z-z_0}{t-z_0}\right|^{n+1} \frac{|dt|}{(|t|-|z|)}$$

$$\le \frac{M}{2\pi}\left(\frac{r}{R}\right)^{n+1}\frac{2\pi R}{(R-r)} \to 0 \text{ as } n \to \infty$$

Hence $\lim\limits_{n\to\infty} \rho_n(z) = 0$

$$|\rho_n'(z)| \le \frac{1}{2\pi}\int_{C_2} |f(t)| \left|\frac{t-z_0}{z-z_0}\right|^{n+1} \frac{|dt|}{(|z|-|t|)}$$

$$\le \frac{M'}{2\pi}\left(\frac{R'}{r}\right)^{n+1}\frac{2\pi R'}{(r-R')} \to 0 \text{ as } n \to \infty$$

Hence $\lim\limits_{n\to\infty} \rho_n'(z) = 0$

Making $n \to \infty$ in (1) and noting the above facts

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}.$$

Note

since two integrands $f(z)/(z-z_0)^{n+1}$ and $f(z)/(z-z_0)^{-n+1}$ are analytic througout the angular domain $R' < |z-z_0| < R$ and on its boundary. Any closed contour C_0 around that domain in the positive direction can be used as a path of integration in place of the circular paths C_1 and C_2.

Take C_0 a circle whose equation is

$$R' < |t-z_0| = R_0 < R$$

then

$$a_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(t)\,dt}{(t-z_0)^{n+1}}$$

$$b_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(t)\,dt}{(t-z_0)^{-n+1}} = a_{-n}$$

In this event Laurent series become

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} (z-z_0)^{-n} a_{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=-\infty}^{-1} (z-z_0)^n a_n$$

$$= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \text{ with } a_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(t)\,dt}{(t-z_0)^{n}}$$

<u>Note</u> - Any function defined by following expression

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-z_0)^n \qquad R' < |z-z_0| < R$$

is necessarily identical with the Laurent's series.