Theory of Inference for statement calculus:

It is concerned with inferring conclusion from certain premises. Here the premises are assumed to be true and then following the rules ~~the~~ we find the conclusion (called valid conclusion).

Validity ~~rule~~ using Truth table: We know $A \Rightarrow B$ means B logically follows from A or B is valid conclusion of premise A iff $A \to B$ is a tautology.

Now, if instead of single premise A we have premises, say, $H_1, H_2, -- H_n$ and conclusion is C then the conclusion C will follow from premises logically iff

$$H_1 \wedge H_2 \wedge -- H_n \Rightarrow C \quad -(*)$$

For this follow following steps.

(1) Look for the rows in which $H_1, -- H_n$ are all true. If C is also true then (*) holds.

(2) Look for the rows in which C is F (i.e, C has truth value F). If in such a row atleast one of the H_i is false then (*) holds.

(3) If for all rows in truth table (*) holds then C is a valid conclusion else invalid.

Ex Determine whether the conclusion C is valid in the following, when $H_1, H_2, --$ are the premises.

(a) $H_1: \neg Q$ $H_2: P \to Q$ $C: \neg P$

(b) $H_1: \neg P$ $H_2: P \vee Q$ $C: P \wedge Q$

(c) $H_1: P \to (Q \to R)$ $H_2: P \wedge Q$ $C: R$

(d) H1: P-->(Q-->R) H2: R C: P

(a) $H_1: \neg Q$ $H_2: P \to Q$ $C: \neg P$

		H_1	H_2	C	
P	Q	$\neg Q$	$P \to Q$	$\neg P$	
T	T	F	T	F	✓
T	F	T	F	F	✓
F	T	F	T	T	I.C (inconclusive)
F	F	T	T	T	✓

Hence it is valid conclusion.

(b) $H_1: \neg P$ $H_2: P \vee Q$ $C: P \wedge Q$

		H_1	H_2	C	
P	Q	$\neg P$	$P \vee Q$	$P \wedge Q$	
T	T	F	T	T	I.C.
T	F	F	T	F	✓
F	T	T	T	F	× → Here H_1 & H_2 are T but C is F ∴ Not valid.
F	F	T	F	F	✓

Invalid conclusion.

(c) $H_1: P \to (Q \to R)$ $H_2: P \wedge Q$ $C: R$

				H_1	H_2	C	
P	Q	R	$Q \to R$	$P \to (Q \to R)$	$P \wedge Q$	R	
T	T	T	T	T	T	T	✓
T	T	F	F	F	T	F	✓
T	F	T	T	T	F	T	I.C
T	F	F	T	T	F	F	✓
F	T	T	T	T	F	T	I.C
F	T	F	F	T	F	F	✓
F	F	T	T	T	F	T	I.C
F	F	F	T	T	F	F	✓

∴ Valid conclusion

(d) $H_1: P \to (Q \to R)$ $H_2: R$ $C: P$ → Invalid.

From above table 2nd row: $H_1 \to F$ $H_2 \to F$ $C \to T$ ∴ ~~Invalid~~ I.C.

5th row: $H_1 \to T$ $H_2 \to T$ $C \to F$ Invalid

7th row: $H_1 \to T$ $H_2 \to T$ $C \to F$ Invalid

Rules of Inference

Rule P: A premise may be introduced at any point in the derivation

Rule T: A formula S may be introduced in a derivation if S is tautologically implied by any one or more of the preceding formulas in derivation

Some important Implication & Equivalences

(1) $\neg P \Rightarrow P \to Q$ (2) $Q \Rightarrow P \to Q$.

(3) $\neg(P \to Q) \Rightarrow P$ (4) $\neg(P \to Q) \Rightarrow \neg Q$

(5) $\neg P, P \vee Q \Rightarrow Q$ (disjunctive syllogism).

(6) $P, P \to Q \Rightarrow Q$ (modus ponens)

(7) $\neg Q, P \to Q \Rightarrow \neg P$ (modus tollens)

(8) $P \to Q, Q \to R \Rightarrow P \to R$ (hypothetical Syllogism)

(9) $P \vee Q, P \to R, Q \to R \Rightarrow R$ (dilemma)

(10) $P \to Q \Leftrightarrow \neg P \vee Q$

(11) $\neg(P \to Q) \Leftrightarrow P \wedge \neg Q$

(12) $P \to Q \Leftrightarrow \neg Q \to \neg P$.

(13) $P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$.

(14) $\neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$.

(15) $P \rightleftarrows Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

Ex. Show the validity of following arguments, for which premises are given on the left and conclusion on the right

a) $\neg(P \wedge \neg Q), \neg Q \vee R, \neg R$ $\neg P$

b) $(P \wedge Q) \to R, \neg R \vee S, \neg S$ $\neg P \vee \neg Q$

c) $(P \to Q) \to R, P \wedge S, Q \wedge T$ R

d) $\neg P \vee Q, \neg Q \vee R, R \to S$ $P \to S$

Soln (a) $\overset{H_1}{\neg(P \wedge \neg Q)}, \overset{H_2}{\neg Q \vee R}, \overset{H_3}{\neg R}$ $\neg P$

(1) $\neg R$ (H3) Rule P

OR

(2) $\neg Q \vee R$ (H2) Rule P

(3) $\neg Q$ (1),(2) Rule T

$Q \rightarrow R$

$\sim Q$ (1),(2) Modus Tollens

(4) $\neg(P \wedge \neg Q)$ (H1) Rule P

(5) $\neg P \vee Q$ (4) De Morgans law

(6) $\neg P$ (3),(5) Rule T //

H1 H2 H3 C

(b) $(P \wedge Q) \rightarrow R$, $\neg R \vee S$, $\neg S$ $\neg P \vee \neg Q$

(1) $\neg S$ (H3) Rule P

(2) $\neg R \vee S$ (H2) Rule P (2*) R-->S

(3) $\neg R$ (1),(2), Rule T 'OR' modus tollens

(4) $(P \wedge Q) \rightarrow R$ (H1) Rule P

(5) $\neg(P \wedge Q)$ (3),(4), Modus tollens

(6) $\neg P \vee \neg Q$ (5) De Morgans law //

H1 H2 H3 C

(c) $(P \rightarrow Q) \rightarrow R$, $P \wedge S$, $Q \wedge T$ R.

(1) $P \wedge S$ (H2) Rule P

(2) P (1) Rule T

(3) $Q \wedge T$ (H3) Rule P

(4) Q (3) Rule T

(5) $P \rightarrow Q$ (2),(4), Rule T

(6) $(P \rightarrow Q) \rightarrow R$ (H1) Rule P

(6) R (5),(6), Modus ponens //

H1 H2 H3 C

(d) $\neg P \vee Q$, $\neg Q \vee R$, $R \rightarrow S$ $P \rightarrow S$

(1) $\neg P \vee Q$ (H1) Rule P

(2) $P \rightarrow Q$ (1) Rule T

(3) $\neg Q \vee R$ (H2) Rule P

(4) $Q \rightarrow R$ (3) Rule T

(5) $P \rightarrow R$ (2),(4), Hypothetical syllogism

(6) $R \rightarrow S$ (H3) Rule P

(7) $P \rightarrow S$ (5),(6) Hypothetical syllogism) //

Rule CP: If we can derive S from R and a set of premises, then we can derive $R \to S$ from set of premises alone

This rule is generally applied when conclusion is of the form $R \to S$. In such a case R is taken as an additional premise

Ex See above question (d) part. It can also be done as:

(1)	P	(additional premise) Rule CP
(2)	$\neg P \vee Q$	(H1), Rule P
(3)	Q	(1),(2), Rule T
(4)	$\neg Q \vee R$	(H2), Rule P
(5)	R	(3),(4), Rule T
(6)	$R \to S$	(H3) Rule P
(7)	S	(5),(6), modus ponens
(8)	$P \to S$	(1),(7), Rule T //

Ex $\overset{H1}{P}$, $\overset{H2}{P \to (Q \to (R \wedge S))}$ $\quad$ $\overset{C}{Q \to S}$

(1)	P	(H1) Rule P
(2)	$P \to (Q \to (R \wedge S))$	(H2) Rule P
(3)	$Q \to (R \wedge S)$	(1),(2) modus ponens
(4)	Q	Rule CP → (necessary)
(5)	$R \wedge S$	(3),(4), modus ponens
(6)	S	(5), Rule T
(7)	$Q \to S$	(4),(7), Rule T

Ex $\overset{H1}{P \to Q}$ $\quad$ $\overset{C}{P \to (P \wedge Q)}$

(1)	$P \to Q$	(H1) Rule P
(2)	P	Rule (CP) → (necessary)
(3)	Q	(1),(2), modus ponens
(4)	$P \wedge Q$	(2),(3), Rule T
(5)	$P \to (P \wedge Q)$	(2),(4), Rule T. //

Try it
Ex $P \to (Q \overset{H_1}{\to} R)$, $Q \overset{H_2}{\to} (R \to S) \Rightarrow \overset{C}{P \to (Q \to S)}$

Soln (1) $P \to (Q \to R)$ (H1) Rule P
(2) P Rule CP
(3) $Q \to R$ (1), (2), modus ponens
(4) $Q \to (R \to S)$ (H2) Rule P
(5) $\neg Q \vee (\neg R \vee S)$ (4), Rule T
(6) $\neg R \vee (\neg Q \vee S)$ (5), Rule T
(7) $R \to (Q \to S)$ (6) Rule T
(8) $Q \to (Q \to S)$ (3), (7), Hypothetical Syllogism
(9) $Q \to S$ (8) Rule T $(Q \to (Q \to S)$
$\Leftrightarrow \neg Q \vee \neg Q \vee S \Leftrightarrow \neg Q \vee S$
$\Leftrightarrow Q \to S)$
(10) $P \to (Q \to S)$ (2), (9), Rule T //

Indirect method of proof: (or proof by contradiction).
Assume C is false and take ¬C is an additional premise.
If this along with other premises gives a contradiction,
then it means C logically follows from given premises

Ex $\overset{H_1}{R \to \neg Q}$, $\overset{H_2}{R \vee S}$, $\overset{H_3}{S \to \neg Q}$, $\overset{H_4}{P \to Q} \Rightarrow \overset{C}{\neg P}$
(1) $P \to Q$ (H4) Rule P
(2) $S \to \neg Q$ (H3) Rule P
(3) $Q \to \neg S$ (2), Contrapositive $(P \to Q \Leftrightarrow \neg Q \to \neg P)$
(4) $P \to \neg S$ (1), (3), hypothetical syllogism
(5) $R \vee S$ (H2) Rule P
(6) $\neg S \to R$ (5), Rule T
(7) $P \to R$ (4), (6), hypothetical syllogism
(8) $R \to \neg Q$ (H1), Rule P
(9) $P \to \neg Q$ (7), (8), hypothetical syllogism → Now what ??
(10) P Method of indirect: Assuming $\neg C$ is true)
(11) $\neg Q$ (9), (10), modus ponens
(12) Q (1), (10), modus ponens
(13) $Q \wedge \neg Q$ (11), (12) Rule T
↳ = F contradiction ∴ C is valid conclusion.