Topology & Topological Spaces -

Let X be non empty set. A class T of subsets of X is called topology on X if it satisfies the following two conditions ① ϕ and X are subsets of T
② the union of every class of sets in T is a set in T
③ the intersection of every finite class of sets in T is a set in T.

A topology on X is thus a class of subsets of X which is closed under the formation of arbitrary unions and intersections. A topological space consists of a non empty set X and a topology T on X. The sets in the class T are called the open sets of the topological space (X, T) and the object of X are called its points.
The same set X may have different topologies.

Example ①

Let X be any metric space and let topology be class of all subsets of X which are open. This is called usual topology on the metric space and we say that these sets are the open sets generated by metric on the space. Metric spaces are most important topological spaces.

Example ② Let X be any non empty set and let the topology be the class of all subsets (power set) of X. This is called descrete topology on X ~~and~~ and any topological space whose topology is the descrete topology is called a discrete space

Example ③

When the topology T on X consists only of ϕ and X then topology is called as the indiscrete topology on X. This topology is induced by the indiscrete pseudo metric on X. From the point of view of convergence of sequence this topology is not discriminating in that every sequence converges to every point since there is only one non empty set.

Example ④ Let X be any infinite set. A subset A of X is said to be [illegible]

Let ~~topology~~ $\mathcal{T}$ consists of all cofinite subsets of X and the empty set, then $\mathcal{T}$ is a topology on X, called cofinite topology or T_1 topology

Example (5) A topology is there which is known as the semi open interval topology. A subset U is said to be open in this topology if for every $x \in U$ there exists $r > 0$ s.t. semi open intervals $[x, x+r)$ is contained in U. This topology is also known as the upper limit topology since a sequence $\{x_n\}$ converges to y in the semi-open interval topology iff it converges to y from the right in the usual topology.

Example (6) Another topology on R is a scattering topology. In this topology the open sets are of the form $A \cup B$ where A is an open subset of R in the usual topology and B is any subset of the irrational number. The resulting topological space is called as the scattered line.

Example (7) A topological space is called as Sierpinski space if it contains only two elements $\{a, b\}$ where $a \neq b$. A topology consists of ϕ, $\{a\}$, and $\{a, b\}$.

Example (8) Let a set X be linearly ordered set by $\leq$. A subset A of X is said to be open if for each $x \in A$ $\exists$ $a, b \in X$ s.t. $a < x < b$ and the interval (a, b) is contained in A. The collection of all open sets is a topology. It is called as the order topology induced by the order $\leq$. For the real line R the topology induced by natural ordering coincides with the usual topology.

The topology generated by the family of intervals of form $(a, \infty) = \{x \in R : x > a\}$ is called right hand topology.

The topology generated by the family of intervals of form $(-\infty, a) = \{x \in R : x > a\}$ is called left hand topology.

(a) Give (i) examples of a proper non empty subsets of a topological space such that it is both open and closed and prove that your assertion.

Soln.

In every descrete topological space (X, T), consisting of more than one element, every proper subset of X is both open and closed

Consider the following topological spaces in which every proper subset of X is both open and closed.

eg. (1) Let $X = \{a, b, c\}$ $T = \{\phi, X, \{a\}, \{b, c\}\}$

Then (X, T) is topological space. Evidently $\{a\}$ $\{b, c\}$ are T open sets so that

$X - \{a\}$, $X - \{b c\}$, are T closed sets i.e $\{b, c\}$ $\{a\}$ are closed.

$\therefore$ $\{a\}$, $\{b, c\}$ are both open as well as closed.

(2) Suppose $X = \{a, b, c\}$ and

$T = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}$

Then T is a topology on X. Also each proper subset of X is open as well as closed.

Example (2) Give an example to show that the union of an infinite collection of closed sets in a topological space is not closed.

Soln.

Let T be denote the usual topology on R. Let

$$F_n = [0, \tfrac{n}{n+1}] \quad \forall n \in N$$

then F_n is closed set

$$\bigcup_{n=1}^{\infty} F_n = [0, \tfrac{1}{2}] \cup [0, \tfrac{2}{3}] \cup \cdots \cup [0, 1)$$

$= [0, 1)$ = semi open set $\neq$ A closed set

$\therefore$ $\bigcup_{n=1}^{\infty} F_n$ is not closed set though each F_n is closed

Theorem (1) A subset A of topological space (X,T) is closed iff $\bar{A} = A$

Proof: Let (X,T) be topological space and $A \subset X$. Firstly we shall establish the following lemmas:

(i) $\bar{A}$ is closed and $A \subset \bar{A}$

(ii) $F \supset A$, F is closed $\Rightarrow F \supset \bar{A}$

By def. $\bar{A} = \cap\{F \subset X : F \text{ is closed } F \supset A\}$ — (1)

(i) Since an arbitrary intersection of closed sets is closed and hence $\bar{A}$ is closed and by (1) $A \subset \bar{A}$

(ii) Suppose $F \subset X$ is closed and ~~F ⊃ A~~ $F \supset A$
$\therefore$ F is among those members whose intersection is $\bar{A}$ and hence $F \supset \bar{A}$.

Now we come to proof of actual theorem
Let $A = \bar{A}$ we claim A is closed. By (i) $\bar{A}$ is closed
$\bar{A}$ is closed, $\bar{A} = A \Rightarrow A$ is closed

Conversely let A is closed. To prove that $A = \bar{A}$
By the lemma (i) $A \subset \bar{A}$ — (2)
By the lemma (ii) F is closed $F \supset A \Rightarrow F \supset \bar{A}$ — (3)
Since $A \supset A$ trivially set A

In particular (3) gives
A is closed $A \supset A \Rightarrow A \supset \bar{A}$ — (4)
from (2) & (4)

$$A = \bar{A}$$

Theorem (2) To show that A^o is largest open subset of A.

Proof: Let (X,T) be topological space and $A \subset X$
By def. $A^o = \cup\{G : G \subset A \text{ and } G \text{ is open}\}$.
An arbitrary union of open sets is open hence A^o is open