UNIT - III

Dt. 04-09-23

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Uniform convergence of Sequences and series of functions

* Sequence and series of function ~

If $f : N \to X$ is a function defined by $f(n) = U_n(x), \forall x \in X, \forall n \in N$.

Then

→ $\{u_n(x)\}_{n \in N}$ is a sequence of fuctions.

$\{u_1(x), u_2(x), \dots\dots U_n(x), \dots\}$

→ $\sum_{n=1}^{\infty} U_n(x)$ is a series of function.

* Example: $\left\{\frac{n^2 x}{1+n^4x^2}\right\}$ is a sequence of function.

$\sum_{n=1}^{\infty} \frac{n^3 x}{1+n^6 x^2}$ is a series of function.

* Pointwise Convergence of Sequence of function

Let sequence of $\{f_n(x)\}$ be a sequence of function with domain S.

Let $c \in S$ → interval

$\forall \ c \in S \quad \{f_n(c)\} = f_1(c), f_2(c), \dots\dots f_n(c), \dots$

$\forall c_1 \in S, \{f_n(c_1)\} = f_1(c_1), f_2(c_1), \dots$

$\{|x_n - l| < \varepsilon, \ \forall\ n \geq n_0\}$ $\{x_n \to l\}$

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$\forall C_2 \in S, \{f_n(C_2)\} = f_1(C_2), f_2(C_2), \ldots\ldots$

$\vdots$

$\forall\ c \in S$, if $f_n(c)$ converges to $f(c)$

i.e., $\lim f_n(c) = f(c)$.

Then f is called pointwise limit of $\{f_n(x)\}$. defined in S.

⇛ **Definition** ~ A function f is the pointwise limit of (pointwise convergent) a sequence of functions $\{f_n(x)\}$ defined in S if for every $c \in S$ and every $\varepsilon > 0$, there exist a positive integer m, such that

$$\boxed{|f_n(c) - f(c)| < \varepsilon, \quad \forall\ n \geq m}$$

depend on both c, ε for pointwise

Note - A sequence which is not a pointwise convergent can not be uniformly convergent.

non-pointwise convergent ⇒ non uniform convergent

In other cases depend on ε only.

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* uniform convergence of sequence of functions:

A sequence of functions $\{f_n(x)\}$ defined in S is said to be uniformly convergent to a function f defined in S if for every $\varepsilon > 0$ there exist a positive integer m such that

$$|f_n(x) - f(x)| < \varepsilon, \ \forall \ n \geq m - \varepsilon \text{ and } \forall x \in S.$$

* Theorem - Every uniformly convergent sequence of function is pointwise convergent but converge need not be true.

Proof - let $\{f_n(x)\}$ be a sequence of functions defined in $a \leq x \leq b$. Suppose $\{f_n(x)\}$ converges uniformly to $f(x)$ in $a \leq x \leq b$.

By definition, for every $\varepsilon > 0$ there exist a positive integer m, such that

$$|f_n(x) - f(x)| < \varepsilon, \ \forall n \geq m \text{ and } \forall x \in [a,b] \quad \text{--- (1)}$$

let $c \in [a, b]$
it follows that, from (1)

$$|f_n(c) - f(c)| < \varepsilon, \ \forall \ n \geq m$$

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or $\lim f_n(c) = f(c)$.

Hence, $\{f_n(x)\}$ is pointwise convergent and it converges to $f(x)$ in $[a, b]$.
This implies that uniform limit is same as pointwise limit.

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* Conversely

we shall show that every pointwise convergent sequence of functions is not uniformly convergent.

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad \forall\, x \in R.$$

At $x = 0$, $f_n(0) = 0$

$$\lim f_n(0) = 0$$

For $x \neq 0$ $\quad f(n) = \dfrac{nx}{1+n^2x^2}$

$$\lim f_n(x) = \lim \frac{nx}{1+n^2x^2}$$

$$= \lim \frac{x/n}{x^2 + \frac{1}{n^2}}$$

$$= 0$$

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$\lim f_n(x) = 0 = f(x)$

Therefore, $\{f_n(x)\}$ converges pointwise to $f(x)$

where $f(x) = 0$, $\forall x \in R$.

Suppose $\{f_n(x)\}$ is uniformly converges in $[a, b]$, including point $x = 0$.

$\Rightarrow$ uniform limit = pointwise limit

For every $\varepsilon > 0$, there exist a positive integer m, such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq m.$$

$$\left| \frac{nx}{1+n^2x^2} - 0 \right| < \varepsilon \quad \forall n \geq m$$

$$\left| \frac{nx}{1+n^2x^2} \right| < \varepsilon, \quad \forall n \geq m$$

Taking $\varepsilon = \frac{1}{4}$

choose a positive integer k, such that $k \geq m$ and $\frac{1}{k} \in (a, b)$

Taking $n = k$ and $x = \frac{1}{k}$. By above

equality we have $\left| \frac{nx}{1+n^2x^2} \right| = \left| \frac{k \cdot \frac{1}{k}}{1 + k^2 \cdot \frac{1}{k^2}} \right|$

$= \frac{1}{2} > \frac{1}{4}$

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which is contradiction. Hence, converge of this theorem is not true.

Ques Discuss the uniform convergence of the sequence of functions $\{f_n(x)\}$ where $f_n(x) = \frac{n}{x+n}$, $\forall\ x \in [0, \infty)$

At $x = 0$,

$$f_n(x) = 1$$

$$\lim f_n(x) = f(x) = 1$$

for $x \in (0, \infty)$

$$f(x) = \lim f_n(x) = \lim \frac{n}{x+n} = 1$$

$$0 < \varepsilon < 1$$

$$|f_n(x) - f(x)| = \left| \frac{n}{x+n} - 1 \right|$$

$$= \left| \frac{x}{x+n} \right| < \varepsilon$$

if $n > x\left(\frac{1}{\varepsilon} - 1\right)$

$$m = x\left(\frac{1}{\varepsilon} - 1\right)$$

$\therefore$ $\exists$ no positive integer m such that,

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$|f_n(x) - f(x)| < \varepsilon, \forall n \geq m, \forall x \in [0, \infty)$

Hence the given sequence is not uniformly convergent.

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* Uniform Convergence of Series

Let $\{u_n(x)\}$ be a sequence of functions defined in $a \leq x \leq b$.

Consider a infinite series

$u_1(x) + u_2(x) + \cdots\cdots u_n(x) + \cdots\cdots, \forall x \in [a,b]$

$$\boxed{\text{or } S_n = \sum_{r=1}^{n} u_r(x), \forall x \in [a,b]}$$

we say that the series

$\sum_{n=1}^{\infty} u_n(x)$ converges uniformly

to $S(x)$ in $a \leq x \leq b$ if for every ε greater than zero $\varepsilon > 0$, $\exists$ a positive integer m, depending on ε only and not on x.

Such that

$|S_n(x) - S(x)| < \varepsilon, \forall n \geq m, \forall x \in [a,b]$

* Cauchy Criteria for uniform converge of series

Theorem:
A necessary and sufficient condition for a uniform convergence of series in $a \le x \le b$ is that for $\varepsilon > 0$ there exist a positive integer m depending on ε only and not on x such that

$$(*) \quad |u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)| < \varepsilon$$

$$\forall\, n \geq m,\ \forall\, x \in [a,b]$$

and for all positive integeral value of p.

Proof ~

→ Necessary Part : suppose series $\sum_{n=1}^{\infty} u_n(x)$ is a uniformly convergent and its sum is $S(x)$, $\forall\, x \in [a,b]$

Write $S_n(x) = \sum_{r=1}^{n} u_r(x)$.

For $\varepsilon > 0$, there exist a positive integer m, depending on ε only and not on x. Such that

$$|S_n(x) - S(x)| < \frac{\varepsilon}{2}, \forall\, n \geq m \quad \text{--- (1)}$$

from (1) it follows that

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$$|S_{n+p}(x) - S(x)| < \frac{\varepsilon}{2}, \quad \forall n \geq m, \; \forall x \in [a,b], \; \forall p > 0 \quad \text{—} (2)$$

Now,

$$= |S_{n+p}(x) - S_n(x)| = |S_{n+p}(x) - S(x) + S(x) - S_n(x)|$$

$$\leq |S_{n+p}(x) - S(x)| + |S_n(x) - S(x)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad [\text{from eq } (1) \;\&\; (2)]$$

$$< \varepsilon$$

$$\boxed{|S_{n+p}(x) - S_n(x)| < \varepsilon, \quad \forall n \geq m, \; \forall x \in [a,b] \; \forall p > 0}$$

$$|U_{n+1}(x) + U_{n+2}(x) + \dots \dots + U_{n+p}(x)| < \varepsilon \; [\forall n \geq m, \forall x \in [a,b]$$

→ Sufficient Part

Suppose condition (*) holds, then by Cauchy general principle of convergence of series $\sum_{n=1}^{\infty} U_n(x)$ converges.

Let the sum be $S(x)$.

Suppose for $\varepsilon > 0$ there exist a positive integer m ~~depending~~ independent of x. Such that

$$|S_{n+p}(x) - S_n(x)| < \frac{\varepsilon}{2} \quad \forall n \geq m, \; \forall x \in [a,b], \; \forall p > 0.$$

It follows that

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$$S_n(x) - \frac{\varepsilon}{2} < S_{n+p}(x) < S_n(x) + \frac{\varepsilon}{2}.$$

Keeping n fixed, let $p \to \infty$

$$S_{n+p} \to S$$

Then,

$$S_n(x) - \frac{\varepsilon}{2} < S(x) < S_n(x) + \frac{\varepsilon}{2}$$

or

$$|S_n(x) - S(x)| < \frac{\varepsilon}{2} = \varepsilon \quad \forall\, n \geq m$$
$$\forall\, x \in [a,b].$$

Hence the series $\sum_{n=1}^{\infty} U_n(x)$ is uniformly convergent.

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