Dt. 11-09-23

Pg.

* Weierstrass M- test ~ (Series)

Theorem ~ A series $\sum_{n=1}^{\infty} U_n(x)$ converges uniformly in $a \leq x \leq b$ if there exist a convergent series $\sum_{n=1}^{\infty} M_n$ of positive constant such that

$$|U_n(x)| \leq M_n \;, \forall\, x \in [a, b] \;, \forall\, n \geq 1 .$$

Proof: Since $\sum_{n=1}^{\infty} M_n$ is convergent, therefore for $\varepsilon > 0$, $\exists$ a positive integer N such that

$$\sum_{n \in N+1}^{N+p} M_n < \varepsilon \;, \forall \text{ positive integral value of } p . \quad \text{--- (1)}$$

It is given that $|U_n(x)| \leq M_n, \forall\, n \in N, x \in [a, b]$ — (2)

Now, $\left|\sum_{n=N+1}^{N+p} U_n(x)\right| \leq \sum_{n=N+1}^{N+p} |U_n(x)| \leq$

$$\sum_{n=N+1}^{N+p} M_n < \varepsilon \text{ (By eq. (2))}$$

$$\left|\sum_{n=N+1}^{N+p} U_n(x)\right| < \varepsilon \;, \forall\, x \in [a, b] \;, \forall \text{ positive integral value of } p .$$

As N is a positive integer independent of x, then by Cauchy general principle of uniform convergence.

Dt. ___
Pg. ___

The series $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent in $[a,b]$.

* M_n - Test (Sequence)

Theorem - Let $\{f_n\}$ be a sequence of functions defined in $[a, b]$.
Let $\lim f_n(x) = f(x) \quad \forall \; x \in [a, b]$

Set $M_n = \sup \{ |f_n(x) - f(x)| : x \in [a,b] \}$

Then the sequence of $\{f_n(x)\}$ converges uniformly to f in $[a, b]$ iff $M_n \to 0$ as $n \to \infty$.

→ Necessary Part

Proof: Suppose sequence $\{f_n(x)\}$ converges uniformly to $f(x)$ in $[a, b]$, therefore $\varepsilon > 0$ there exist a positive integer m independent of x, such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall \; n \geq m, \; \forall \; x \in [a, b]$$

It follows that

$$\sup \{ |f_n(x) - f(x)| : x \in [a,b] \} < \varepsilon, \quad \forall \; n \geq m, \; \forall \; x \in [a,b]$$

or,

$$M_n < \varepsilon, \quad \forall \; n \geq m.$$

or,

$$M_n \to 0, \quad \text{as } n \to \infty$$

Dt. ________

Pg. ________

→ Sufficient Part

Conversely - Let $M_n \to 0$ as $n \to \infty$
for $\varepsilon > 0$, $\exists$ a positive integer m
such that

$$M_n < \varepsilon \quad , \forall n \geq m$$

$$\Rightarrow \sup \{ |f_n(x) - f(x)| : x \in [a,b] \} < \varepsilon , \quad \forall n \geq m.$$

It follows that.

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall x \in [a,b], \forall n \geq m.$$

Hence, by definition of uniform convergence, the $\{f_n(x)\}$ converges uniformly to f in $[a, b]$.

Ques Discuss the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{1}{1+n^4 x^2}$, $\forall x \in [1, \infty)$.

By comparing with $\sum_{n=1}^{\infty} U_n(x)$

$$U_n(x) = \frac{1}{1+n^4 \cdot x^2} \leq \frac{1}{1+n^4} \leq \frac{1}{n^4} = M_n$$

$\sum \frac{1}{n^p}$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \qquad p = 4 > 1.$$

Hence by weierstrass M- test the given series is uniformly convergent in $[1,\infty)$.

$p>1$ convergent

$p<1$ divergent

Dt. 12-09-23

Pg.

Ques. Discuss the uniform convergence of the sequence of $\{f_n(x)\}$ where $f_n(x) = nx(1-x)^n$ $\forall x \in [0,1]$.

At $x = 0$, $f_n(0) = 0$

$$f(x) = \lim f_n(0) = 0$$

At $x = 0$

$$f_n(x) - f(x) = 0$$

for $0 < x < 1$

$$f_n(x) = nx(1-x)^n$$

$$= \frac{nx}{(1-x)^{-n}}$$

$\left(\frac{1}{2}\right)^{-\infty} = \frac{1}{2^{-\infty}} = \infty$

$$\lim f_n(x) = \lim \frac{x}{-(1-x)^{-n}\log(1-x)}$$

$$= \lim \frac{x(1-x)^n}{-\log(1-x)}$$

$$= 0$$

$f(x) = 0$, $\forall x \in [0,1]$

$M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$

Let $y = nx(1-x)^n$

$$\frac{dy}{dx} = n\left[x \cdot n(1-x)^{n-1} \cdot (-1) + (1-x)^n\right]$$

$$= n(1-x)^n\left[\frac{-nx}{1-x} + 1\right]$$

Dt. ______

Pg. ______

$$-nx + 1 - x = 0$$
$$-x(n+1) = -1$$

$$x = 1 \qquad x = \frac{1}{n+1}.$$

$$y_{max} = n \cdot \frac{1}{n+1}\left[1 - \frac{1}{n+1}\right]^n$$
$$= \frac{n}{n+1}\left[\frac{n}{n+1}\right]^n = \frac{n^{n+1}}{(n+1)^{n+1}}$$

$$M_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}$$

$$\lim M_n = \frac{1}{e} \neq 0.$$

Hence, by M_n test the given sequence is not uniformly convergent in $[0,1]$.

Ques Discuss the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$, $\forall x \in [0,1]$.

By comparing with $\sum_{n=1}^{\infty} U_n(x)$.

$$U_n(x) = \frac{x}{n(1+nx^2)} \leq \frac{1}{n(1+n)} \leq \frac{1}{n^2} = M_n$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad p = 2 > 1.$$

Hence by M-test series is uniformly convergent.

Dt.

Pg.

$$y = \frac{x}{n(1+nx^2)}$$

$$\frac{dy}{dx} = \frac{(1+nx^2)\times 1 - x\times 2nx}{n(1+nx^2)^2} \Rightarrow 0$$

$$1 + nx^2 - 2nx^2 = 0$$

$$1 - nx^2 = 0$$

$$1 = nx^2$$

$$x = \pm\frac{1}{\sqrt{n}}$$

$$\frac{d^2y}{dx^2} = \frac{1-nx^2}{n(1+nx^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{n}\left[\frac{(1+nx^2)^2 \times -2nx - (1-nx^2)\times 2(1+nx^2)\times 2nx}{(1+nx^2)^4}\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{n}\left[\frac{\left(1+n\times\frac{1}{n}\right)^2 \times -2\times n\times\frac{1}{\sqrt{n}} - \left(1-n\times\frac{1}{n}\right)^2 \cdots}{\left(1+n\times\frac{1}{n}\right)^4}\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{n}\left[\frac{4\times -2\sqrt{n}}{16}\right] = \frac{-1}{2\sqrt{n}} \text{ (max value)}$$

$$y = \frac{x}{n(1+nx^2)}$$

$$y_{max} = \frac{1/\sqrt{n}}{n\left(1+n\times\frac{1}{n}\right)} = \frac{1}{2n^{3/2}} \Rightarrow M_n$$

$$\lim_{n\to\infty} M_n \Rightarrow \lim_{n\to\infty}\frac{1}{2n^{3/2}} \Rightarrow 0$$

Hence, by M_n - Test the series is uniform convergent on $[0,1]$.

Ques Show that the series $\Sigma r^n \sin n\theta$ is uniformly convergent for all real value of θ and $0 < r < 1$.

$$|r^n \sin n\theta| \le \overset{M_n}{r^n}, \quad \forall \theta$$

$$\Sigma r^n$$

$$\lim r^n = 0 \quad \text{Since } \Sigma m \text{ is convergent}$$

Hence by M_n-test the given series is uniformly convergent.

14-09-23

Theorem - The sum of uniformly convergent series of continous function is a continous function.

Proof - Let $\sum_{n=1}^{\infty} U_n(x)$ be a uniformly convergent series of continous functions $U_n(x)$, $(\forall n \ge 1)$ in $[a,b]$.

Let sum of series $\sum_{n=1}^{\infty} U_n(x)$ be S.

write $S_n(x) = \sum_{r=1}^{n} U_r(x)$

$\because$ $\Sigma U_n(x)$ is uniformly convergent in $[a,b]$

therefore for every $\varepsilon > 0$, $\exists$ a positive integer m depending on ε only and not on x, such that

$$|S(x) - S_n(x)| < \frac{\varepsilon}{3} \quad \forall n \ge m \quad \forall x \in [a,b] \qquad \text{—— (1)}$$

Dt. ________

Pg. ________

Let c be a point in (a, b)

Now,

$$|S(x) - S(c)| = |S(x) - S_n(x) + S_n(x) - S_m(c) + S_m(c) - S(c)|$$

$$\leq |S(x) - S_n(x)| + |S_n(x) - S_m(c)| + |S_m(c) - S(c)|$$

from ①,

$$|S(x) - S(c)| < \frac{\varepsilon}{3} + |S_n(x) - S_m(c)| + \frac{\varepsilon}{3}$$

$$< \frac{2\varepsilon}{3} + |S_n(x) - S_m(c)| \quad \text{—} ②$$

Since $S_m(x)$ is the sum of m continous function in $[a, b]$. Therefore it is also continous in $[a, b]$.

For $\varepsilon > 0$, $\exists$ a positive $\delta > 0$ such that

$$|S_m(x) - S_m(c)| < \frac{\varepsilon}{3}, \text{ whenever } |x - c| < \delta \quad \text{—} ③$$

Combining ② and ③; we get

$$|S(x) - S(c)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$= \frac{3\varepsilon}{3} \cancel{<} \varepsilon$$

$$|S(x) - S(c)| < \varepsilon, \text{ whenever } |x - c| < \delta$$

Dt.
Pg.

Hence, the sum of uniformly convergent series of continuous function is continuous.

* Term by Term differentiation ~

If

i) the series $\sum U_n(x)$ converges in $a \leq x \leq b$ and its sum is $S(x)$

ii) each $U_n'(x)$ exist and is continuous in $a \leq x \leq b$

iii) the series $\sum U_n'(x)$ converges uniformly in $a \leq x \leq b$ and if ~~some~~ sum is $\sigma(x)$, then

$$\sigma(x) = S'(x)$$