* Term by term Integration ~
If
i) the series $\sum U_n(x)$ converges uniformly in $a \le x \le b$.

ii) each $U_n(x)$ is bounded and integrable in $a \le x \le b$ then
$$\int_a^b \left(\sum_{n=1}^{\infty} U_n(x) \right) dx = \sum_{n=1}^{\infty} \int_a^b U_n(x)\, dx$$

* Abel's Test ~~for uniform convergence~~ test

If the series $\sum_{n=1}^{\infty} a_n$ is convergent and the sequence $\{V_n\}$ is monotonic that tend to a finite limit then the series $\sum_{n=1}^{\infty} a_n V_n$ is convergent.

* Dirichlet's Test ~ If $\sum a_n$ is bounded and $\{v_n\}$ is monotonic sequence converging to zero, then $\sum a_n v_n$ is convergent.
OR in other words
proof A series $\sum U_n(x) V_n(x)$ converges uniformly in $[a,b]$ if:
i) the $\{V_n(x)\}$ is monotonic decreasing and uniformly converging to zero in $[a,b]$.
ii) Series $\sum U_n(x)$ is bounded i.e,
$$|S_n(x)| = \left| \sum_{r=1}^{n} U_r(x) \right| \le k, \ \forall x \in [a,b].$$

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* Abel's test for uniform convergence test

If $\sum_{n=1} U_n(x)$

i) series converges uniformly in $a \le x \le b$

ii) the sequence of $\{v_n(x)\}$ is monotonic for every fixed x in $[a,b]$ and is uniformly bounded in $[a,b]$, then the series $\sum U_n(x) V_n(x)$ is uniformly convergent in $[a,b]$

Pf: write ~~$p r_n(x)$~~

$$p^{r_n(x)} = \sum_{r=n+1}^{n+p} u_r(x)$$

and ~~$P R_n(x)$~~

$$p^{R_n(x)} = \sum_{r=n+1}^{n+p} u_r(x) V_r(x)$$

Now,

$$p^{R_n(x)} = \sum_{r=n+1}^{n+p} U_r(x) V_r(x)$$

$$= U_{n+1}(x) V_{n+1}(x) + U_{n+2}(x) + V_{n+2}(x) + \cdots - - - + U_{n+p}(x) V_{n+p}(x)$$

1^{r_n} & $p = 1$

$$= p^{R_n(x)} = 1^{r_n(x)} \cdot V_{n+1}(x) + \left(2^{r_n(x)} - 1^{r_n(x)}\right) V_{n+2}(x) + \cdots\cdots + \left(p^{r_n(x)} - p_{n-1}^{r_n(x)}\right) V_{n+p}(x)$$

$$= pR_n(x) = 1 r_n(x) \left(V_{n+1}(x) - V_{n+2}(x)\right) + 2 r_n(x) \left(V_{n+2}(x) - V_{n+3}(x)\right) + \cdots\cdots + p-1\, r_n(x) \left(V_{n+p-1}(x) - V_{n+p}(x)\right) + p r_n(x) V_{n+p}(x) \quad \text{--- (1)}$$

Next, since the series $\sum U_n(x)$ converges uniformly in $[a,b]$ therefore for $\varepsilon > 0$, $\exists$ a positive integer m depending on

ε only and not on x; such that

$$\sum_{\nu=n+1}^{n+p} U_\nu(x) = \left| {}_p R_n(x) \right| < \frac{\varepsilon}{3k}, \ \forall\, n \geq m, \ \forall\, x \in [a,b], \ \forall\, p > 0 \quad \text{(2)}$$

where k is a positive number such that
$|V_n(x)| < k, \ \forall\, n, \ \forall\, x \in [a,b]$ — (3)

All the expressions $V_{n+1}(x) - V_{n+2}(x)$, $V_{n+2}(x) - V_{n+3}(x)$, ..., $V_{n+p-1}(x) - V_{n+p}(x)$ have the same signs.
From (1), (2) and (3), we have

$$\left| {}_p R_n(x) \right| < \frac{\varepsilon}{3k}\left[\left| V_{n+1}(x) - V_{n+2}(x) \right| + \left| V_{n+p}(x) \right| \right.$$

$$< \frac{\varepsilon}{3k}\left[|V_{n+1}(x)| + |V_{n+2}(x)| + |V_{n+p}(x)| \right.$$

$$< \frac{\varepsilon}{3k} \cdot 3k = \varepsilon$$

or
$\left| {}_p R_n(x) \right| < \varepsilon, \ \forall\, n \geq m, \ \forall\, x \in [a,b], \ \forall\, p > 0$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} U_n(x)\, V_n(x)$ converges uniformly in $[a,b]$.

Proof: Dirichlet's Test

Write $S_n(x) = \sum_{r=1}^{n} U_r(x)$, then $\sum_{n=1}^{\infty} U_n(x) V_n(x)$

converges uniformly in $[a, b]$.

Now, we define

$$pR_n(x) = \sum_{r=n+1}^{n+p} U_r(x)\, V_r(x)$$

Next,

$$pR_n(x) = \sum_{r=n+1}^{n+p} U_r(x)\, V_r(x)$$

$$= U_{n+1}(x) V_{n+1}(x) + U_{n+2}(x) V_{n+2}(x) + U_{n+3}(x) V_{n+3}(x) + \ldots\ldots U_{n+p-1}(x) V_{n+p-1}(x) + U_{n+p}(x) V_{n+p}(x)$$

$$= (S_{n+1}(x) - S_n(x)) V_{n+1}(x) + (S_{n+2}(x) - S_{n+1}(x)) V_{n+2}(x) + \ldots + (S_{n+p-1}(x) - S_{n+p-2}(x)) V_{n+p-1}(x) + (S_{n+p}(x) - S_{n+p-1}(x)) V_{n+p}(x)$$

$$= S_{n+1}(x)(V_{n+1}(x) - V_{n+2}(x)) + S_{n+2}(x)(V_{n+2}(x) - V_{n+3}(x)) + \ldots\ldots + S_{n+p-1}(x)(V_{n+p-1}(x) - V_{n+p}(x)) + S_{n+p}(x)(V_{n+p}(x) - S_n(x) V_{n+1}(x)$$

= By first condition of theorem,

$V_{n+1}(x) - V_{n+2}(x)$, $V_{n+2}(x) - V_{n+3}(x)$, ..., $V_{n+p-1}(x) - V_{n+p}(x)$ are positive.

Then,

$$|pR_n(x)| \le K\Big[|V_{n+1}(x) - V_{n+2}(x)| + |V_{n+2}(x) - V_{n+3}(x)| + \ldots\ldots + |V_{n+p-1}(x) - V_{n+p}(x)| + |V_{n+p}(x)| + |V_{n+1}(x)|$$

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$$= k\left[V_{n+1}(x) - V_{n+2}(x) + \cdots + V_{n+p}(x) + V_{n+1}(x)\right]$$

$$= k\left[2V_{n+1}(x)\right]$$

or

$$|{}_pR_n(x)| \le k\left[2V_{n+1}(x)\right]$$
$$\le 2k\left[V_{n+1}(x)\right] \quad \text{—— (1)}$$

Since $\{V_n(x)\}$ is positive and converging uniformly to zero, therefore for $\frac{\varepsilon}{2k} > 0$,

$\exists$ a positive integer m depending on ε only not on x, such that

$$V_{n+1}(x) < \frac{\varepsilon}{2k}, \ \forall\, n \ge m, \ \forall\, x \in [a, b] \quad \text{—— (2)}$$

from (1) and (2),

$$|{}_pR_n(x)| < 2k \cdot \frac{\varepsilon}{2k} = \varepsilon, \quad \forall\, n \ge m, \forall x \in [a,b], \ p > 0$$

Hence,

$\Sigma\, U_n(x) V_n(x)$ converges uniformly in closed interval $[a, b]$.

montonic Dec: $V_n(x) \geq V_{n+1}(x)$

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Ques: Test the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot x^n}{n^2}$ in $0 \leq x \leq 1$.

Let $U_n(x) = \frac{(-1)^{n-1}}{n^2}$ and $V_n(x) = x^n$

By Leibnitz test, the series $\sum U_n(x)$ is convergent.

Also,

$$|V_n(x)| = |x^n| \leq 1, \forall x \in [0,1]$$

$$V_n = x^n = 1/2^n$$
$$V_{n+1} = x^{n+1} = 1/2^{n+1}$$

$$V_n(x) \geq V_{n+1}(x)$$

Thus the $\{V_n(x)\}$ is monotonic decreasing and bounded.

Hence by Abel's theorem the given series is uniformly convergent.

Ques: Discuss the uniform convergence of

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n+x^2}\right).$$

Let $U_n(x) = (-1)^{n-1}$ and $V_n(x) = \frac{1}{n+}$

$$V_{n+1} = \frac{1}{n+1+x^2}$$

$$V_n(x) \geq V_{n+1}(x)$$

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and $\lim v_n(x) = 0 \quad , \forall x$

Hence, the $\{v_n(x)\}$ is monotonic decreasing and converges to zero uniformly.

$$= \sum_{n=1}^{\infty} (-1)^{n-1}$$
$$= 1 - 1 + 1 - 1 + \cdots\cdots$$

$S_{2n}(x) = 0$

$S_{2n+1}(x) = 1$

$\Rightarrow \quad |S_n(x)| \leq 1$

$\Rightarrow$ The series $\sum_{n=1}^{\infty} u_n(x)$ is uniformly bounded

Hence, by dirichlet's test the given series is uniformly convergent.

Polynomial

A polynomial is a function P on set of real number $\mathbb{R}$ defined by

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots\cdots + a_{n-1}x + a_n, \quad a_0 \neq 0$$

where $a_0, a_1, a_2, \cdots a_n$ are real constants.

If n is a positive integer then we say that $P(x)$ is a polynomial of degree n.

write

$$p_{nr} = \binom{n}{r} x^r (1-x)^{n-r}, \quad \text{where } \binom{n}{r} = {}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}$$

is a polynomial of degree n.

Lemma - for any n and $\forall x \in [0,1]$, $0 \le r \le n$, then

i) $\sum_{r=0}^{n} p_{nr}(x) = 1$

ii) $\sum_{r=0}^{n} r\, p_{nr}(x) = nx$

iii) $\sum_{r=0}^{n} (nx - r)^2\, p_{nr}(x) = nx(1-x)$

Proof:

i) $$\sum_{r=0}^{n} p_{nr}(x) = \sum_{r=0}^{n} \binom{n}{r} x^r (1-x)^{n-r}$$

$$= \sum_{r=0}^{n} {}^nC_r\, x^r (1-x)^{n-r}$$

$$= [x + (1-x)]^n$$

$$= 1$$

ii) $$\sum_{r=0}^{n} r\, p_{nr}(x)$$

$$= \sum_{r=0}^{n} r \cdot {}^nC_r\, x^r (1-x)^{n-r}$$

$$= \sum_{r=0}^{n} r \cdot \frac{\lfloor n}{\lfloor r \lfloor n-r} \cdot x^r (1-x)^{n-r}$$

$$= \sum_{r=0}^{n} \frac{\lfloor n}{\lfloor r-1 \lfloor n-r} x^r (1-x)^{n-r}$$

$$= nx \sum_{r=0}^{n} \frac{\lfloor n-1}{\lfloor r-1 \lfloor n-r} x^{r-1} (1-x)^{n-r}$$

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$$= nx \sum_{r=0}^{n} \binom{n-1}{r-1} x^{r-1} (1-x)^{n-r}$$

$$= nx \sum_{r=\phi}^{n} \binom{n-1}{r-1} x^{r-1} (1-x)^{n-r}$$

Taking $r-1 = k$

$$= nx \sum_{k=0}^{n} \binom{n-1}{k} x^{k} (1-x)^{n-1-k}$$

$$= nx \left[x + (1-x)\right]^{n-1}$$

$$= nx$$

iii) $\sum_{r=0}^{n} r(r-1)\, p_{nr}(x)$

$$= \sum_{r=0}^{n} r(r-1)\, {}^{n}C_{r}\, x^{r} (1-x)^{n-r}$$

$$= \sum_{r=0}^{n} r(r-1) \frac{\lfloor n}{\lfloor r \lfloor n-r} x^{r} (1-x)^{n-r}$$

$$= \sum_{r=0}^{n} \frac{n(n-1) \lfloor n-2}{\lfloor r-2 \lfloor n-r} x^{r} (1-x)^{n-r}$$

$$= n(n-1)x^2 \sum_{r=0}^{n} \binom{n-2}{r-2} x^{r-2} (1-x)^{n-r}$$

~~Taking $r-2 = k$~~

$$n(n-1)\, x^2 \sum_{r=2}^{n} \binom{n-2}{r-2} x^{r-2} (1-x)^{n-r}$$

Dt.

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Putting $r-2=k$

$$= n(n-1)x^2 \sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{n-2-k}$$

$$= n(n-1)x^2 \left[x + (1-x)\right]^{n-2}$$

$$= n(n-1)x^2$$

Now, $\sum_{r=0}^{n} (nx-r)^2 p_{nr}(x)$

$$= n^2x^2 \overbrace{\sum p_{nr}(x)}^{1} - 2nx \overbrace{\sum r\, p_{nr}(x)}^{nx} + \sum r^2 p_{nr}(x)$$

$$= n^2x^2 - 2nx \cdot nx + \sum \left(r(r-1)+r\right) p_{nr}(x)$$

$$= -n^2x^2 + \sum r(r-1) p_{nr}(x) + \sum r\, p_{nr}(x)$$

$$= -n^2x^2 + n(n-1)x^2 + nx$$

$$= nx(1-x)$$

* Bernstein Polynomial

we define Bernstein polynomial defined by

$$B_n(f, x) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{nr}(x)$$

for every function $f : [0,1] \to R$ and $n = 0, 1, 2, \ldots$

Weirstrass approximation theorem ~
If f is continous in $[a,b]$ then $\exists$ a sequence of polynomial for which converges uniformly to f in $[a,b]$.

Proof ~ Without loss of generality, we take $[a,b] = [0,1]$
we shall show that Bernstein Polynomial is the required polynomial for this theorem, we write,

$$p_{nr}(x) = \binom{n}{r} x^r (1-x)^{n-r}$$

$0 \le r \le n$, where $\binom{n}{r} = \dfrac{\lfloor n}{\lfloor r \lfloor n-r}$ is the combination of n objects taking r together for every function.

$$B_n(f,x) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{nr}(x),$$

$f[0,1] = R$ and $n = 0, 1, 2, 3, \ldots\ldots$

Since f is continous in ~~[a,b]~~ $[0,1]$, therefore it is bounded in $[0,1]$ i.e. $\exists$ a positive real constant M such that

$$|f(x)| < \frac{M}{2}, \forall x \in [0,1] \quad \text{--- (1)}$$

Since f is continous in $[0,1]$, therefore it is uniformly continuous in $[0,1]$, i.e., for $\varepsilon > 0$, there exist a number $\delta > 0$

such that ~~x, y~~

$x, y \in [0,1]$,

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$ —— (2)

Now,

$$|f(x) - B_n(f,x)| = \left| f(x) - \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{nr}(x) \right|$$

$$= \left| \sum_{r=0}^{n} f(x) p_{nr}(x) - \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{nr}(x) \right|$$

$$= \cancel{p_{nr}(x)} \left| \sum_{r=0}^{n} \left(f(x) - f\left(\frac{r}{n}\right)\right) p_{nr}(x) \right|$$

$$\leq \sum_{r=0}^{n} \left| f(x) - f\left(\frac{r}{n}\right) \right| p_{nr}(x)$$

$$= \sum_{|x-\frac{r}{n}| \leq \delta} \left| f(x) - f\left(\frac{r}{n}\right)\right| p_{nr}(x) + \sum_{|x-\frac{r}{n}| \leq \delta} \left| f(x) - f\left(\frac{r}{n}\right)\right| p_{nr}(x)$$

$$\leq \frac{\varepsilon}{2} + \sum_{|x-\frac{r}{n}| \geq \delta} f(x) p_{nr}(x) + \sum_{|x-\frac{r}{n}| \geq \delta} f\left(\frac{r}{n}\right) p_{nr}(x) \quad [\text{from } (2)]$$

$$\leq \frac{\varepsilon}{2} + \frac{M}{2} \sum_{|x-\frac{r}{n}| \geq \delta} p_{nr}(x) + \frac{M}{2} \sum_{|x-\frac{r}{n}| \geq \delta} p_{nr}(x) \quad [\text{from } 1]$$

$$= \frac{\varepsilon}{2} + M \sum_{|x-\frac{r}{n}| \geq \delta} p_{nr}(x)$$

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$$< \frac{\varepsilon}{2} + M \sum \frac{(nx-r)^2}{n^2\delta^2} p_{nr}(x)$$

$$= \frac{\varepsilon}{2} + \frac{M}{n^2\delta^2} \sum (nx - r)^2 p_{nr}(x)$$

$$< \frac{\varepsilon}{2} + \frac{M}{n^2\delta^2} nx(1-x)$$

$$= \frac{\varepsilon}{2} + \frac{M}{n\delta^2} x(1-x)$$

$$< \frac{\varepsilon}{2} + \frac{M}{4n\delta^2}$$

$g(x) = x(1-x)$

$g_{max} = \frac{1}{4}$

choose a positive integer m such that

$$\frac{M}{4n\delta^2} < \frac{\varepsilon}{2}$$

Then,

$$|f(x) - B_n(f,x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall n \geq m.$$

Hence, $B_n(f,x)$ converges uniformly to f in $[0,1]$.

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Power Series ~

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

$$\sum_{n=1}^{\infty} a_n x^n$$

Radius of Convergence ~

$$\frac{1}{R} = \lim |a_n|^{1/n} = \lim \left| \frac{a_{n+1}}{a_n} \right|$$

Circle of convergence ~

$|x| \leq R$ (for convergence)

$|x| > R$ (for divergence)

Cauchy n^{th} test ~

If $\lim |U_n|^{1/n} < 1 \rightarrow$ series converges

$> 1 \rightarrow$ series diverges

$= 1 \rightarrow$ test fail

Putting $a_n x^n$ in $\frac{1}{R} = \lim |a_n|^{1/n}$

$$\frac{1}{R} = \lim |a_n x^n|^{1/n}$$

$$\frac{1}{R} = \lim |a_n|^{1/n} \, |x|$$

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$$\frac{|x|}{R} > 1$$

$|x| > 1$ (Divergent)

Ques. Find the radius of convergence of series $\sum_{n=0}^{\infty} \frac{x^n}{\lfloor n}$.

$$\sum a_n x^n$$

$$a_n = \frac{1}{\lfloor n}$$

$$\frac{1}{R} = \lim \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim \left| \frac{\frac{1}{\lfloor n+1}}{\frac{1}{\lfloor n}} \right| = \lim \frac{\cancel{\lfloor n}}{(n+1)\cancel{\lfloor n}}$$

$$= 0$$

$$\frac{1}{R} = 0 \qquad R = \infty.$$

Ques Find the radius of convergence of series $\sum \frac{(\lfloor n)^2 x^n}{\lfloor 2n}$.

$$a_n = \frac{(\lfloor n)^2}{\lfloor 2n}$$