Variance: $Var(X) = E[(X-\mu)^2]$, $\mu = E(X)$.

standard deviation $\sigma_X = \sqrt{Var(X)} = \sqrt{\sigma_X^2}$.

For **discrete random variable X** which takes values $x_1, x_2, \ldots x_n$ with probabilities $f(x_1) \ldots f(x_n)$,

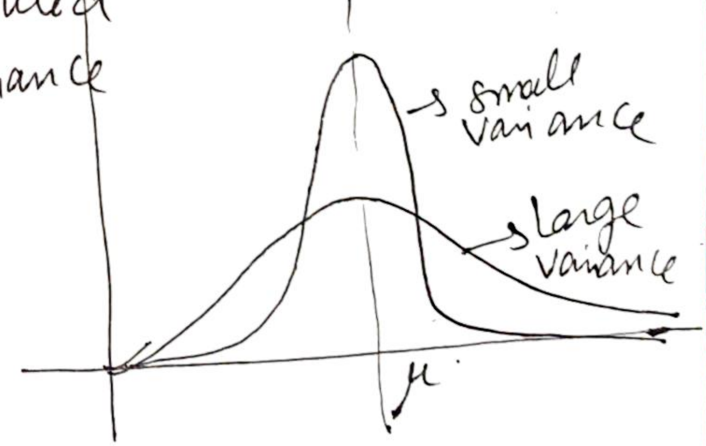
$$\sigma_X^2 = Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i)$$

For **continuous random variable X**: with density function $f(x)$, $-\infty \le f(x) \le \infty$

$$\sigma_X^2 = Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx.$$

Variance is measure of dispersion or scatter.

* If values are concentrated near mean then variance is small (see fig)
* If values are distributed far from mean, variance is large (see fig)



Example: The density function of random variable X is given by $f(x) = \begin{cases} x/2 & 0<x<2 \\ 0 & \text{otherwise} \end{cases}$

find var(X), standard deviation of X.

soln $\sigma_X^2 = Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx = \int_0^2 x \cdot \frac{x}{2}\,dx = \frac{4}{3}$

$\therefore \sigma_X^2 = \int_{-\infty}^{\infty} (x-\frac{4}{3}) f(x)\,dx = \int_0^2 (x-\frac{4}{3}) \frac{x}{2}\,dx$.

$= 2/9$ //

$\therefore \sigma_X = \sqrt{2}/3$ //

Some theorems on variance.

Theorem 1| $\sigma^2 = E[(X-\mu)^2] = E(X^2) - \mu^2 = E(X^2) - (E(X))^2$

Proof: $E[(X-\mu)^2] = \sum_{i=1}^{n} (x_i-\mu)^2 f(x_i)$ X → discrete r.v taking values $x_1, x_2 - x_n$

$= \sum_{i=1}^{n} (x_i^2 - 2x_i\mu + \mu^2) f(x_i)$

$$= \sum_{i=1}^{n} x_i^2 f(x_i) - 2\mu \sum_{i=1}^{n} x_i f(x_i) + \mu^2 \sum_{i=1}^{n} f(x_i)$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2$$

$$= E(X^2) - [E(X)]^2 \quad //$$

Similarly do it for continuous r.v. X.

Theorem 2: $Var(cX) = c^2 Var(X)$, c is constant

Proof: ~~Exax~~ Let X be continuous r.v. with density fn $f(x)$. Then

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

$$\therefore E(cX) = \int_{-\infty}^{\infty} (cx) f(x)\, dx = c \int_{-\infty}^{\infty} x f(x)\, dx = c\mu.$$

$$\therefore Var(cX) = \int_{-\infty}^{\infty} (cx - \mu')^2 f(x)\, dx \quad \text{where } \mu' = E(cX)$$

$$= \int_{-\infty}^{\infty} (cx - c\mu)^2 f(x)\, dx = c^2 \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= c^2 Var(X) \quad //$$

Theorem 3 $E[(X-a)^2]$ is minimum when $\mu = a = E(X)$

Proof: $E[(X-a)^2] = E[\{(X-\mu)+(\mu-a)\}^2]$

$$= E[(X-\mu)^2] - 2(\mu-a)E(X-\mu) + (\mu-a)^2$$

$$= E[(X-\mu)^2] - 0 + (\mu-a)^2 \quad (\because E(X-\mu)=0)$$

$$= E[(X-\mu)^2] + (\mu-a)^2.$$

which will be minimum when $\mu = a$ //.

Theorem 4: If X and Y are independent random variables then $Var(X \pm Y) = Var(X) \pm Var(Y)$.

This can be generalized to more than 2 variables.

Ex Find variance & standard deviation of the sum obtained in tossing a pair of fair dice.

Soln **Method I** $X \to$ sum on no.'s on the two dice

X:	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$$\therefore E(X) = \sum_{x=2}^{12} x f(x) = \frac{252}{36} = 7.$$

Method II Let $X \to$ no. on 1^{st} die
$Y \to$ no. on 2^{nd} die

X & Y are independent

$$\mu_x = E(X) = \sum_{x=1}^{6} x f(x) = \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + \frac{6 \cdot 1}{6} = \frac{7}{2}$$

$$\mu_y = E(Y) = \sum_{y=1}^{6} x f(x) = 7/2$$

$$\therefore E(X+Y) = E(X) + E(Y) = 7.$$

$$\therefore Var(X) = E(X^2) - \mu_x^2 = \Sigma x^2 f(x) - \left(\frac{7}{2}\right)^2$$

$$= \left(1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + \cdots + 6^2 \cdot \frac{1}{6}\right) - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

$$Var(Y) = E(Y^2) - \mu_y^2 = \frac{35}{12}$$

$$\therefore Var(X+Y) = \frac{70}{12} = \frac{35}{6} \text{ //} \qquad \therefore \sigma_{X+Y} = \sqrt{\frac{35}{6}} \text{ //}$$

Ex Let X be a r.v. having density fn

$$f(x) = \begin{cases} \frac{1}{4} & -2 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

. Find 1) $Var(X)$ 2) σ_X.

Soln $$E(X) = \mu = \int_{-2}^{2} x \cdot \frac{1}{4} dx = 0.$$

$$\sigma_X^2 = \int_{-2}^{2} (x-0)^2 \frac{1}{4} dx = \frac{8}{6} = \frac{4}{3} \text{ //} \qquad \therefore \sigma_X = \frac{2}{\sqrt{3}} \text{ //}$$

Ex Let X be a r.v. having density fn

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Soln $\mu = E(X) = \int_0^\infty x \cdot e^{-x} dx = -xe^{-x} - e^{-x} \Big|_0^\infty = 1$

$$\sigma_X^2 = \int_0^\infty (x-1)^2 e^{-x} dx = -(x-1)^2 e^{-x} - 2(x-1)e^{-x} + 2e^{-x} \Big|_0^\infty$$

$$= 1 - 2 + 2 = 1$$

$\therefore \sigma_X = 1$

Ex If a r.v. X is such that $E[(X-1)^2] = 10$ and $E[(X-2)^2] = 6$. Find (a) $E(X)$ (b) σ_X^2 (c) σ_X

Soln (a) $E[(X-1)^2] = 10 \Rightarrow E(X^2) - 2E(X) + 1 = 10$

or, $E(X^2) - 2E(X) = 9$ . . . (1)

$E[(X-2)^2] = 6 \Rightarrow E(X^2) - 4E(X) = 2$

$$\begin{array}{r} E(X^2) - 2E(X) = 9 \\ \underline{-\ E(X^2) \overset{+}{-} 4E(X) = \overset{-}{2}} \\ 2E(X) = 7 \end{array}$$

or, $E(X) = \frac{7}{2}$

(b) Put $E(X) = 7/2$ in (1) gives $E(X^2) = 16$

$\therefore \sigma_X^2 = E(X^2) - (E(X))^2 = 16 - \left(\frac{7}{2}\right)^2 = 15/4$

$\therefore \sigma_X = \sqrt{15}/2$