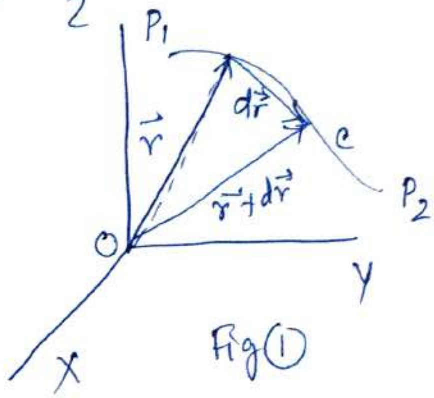
# Work and Energy

## Work

If a force $\vec{F}$ acting on a particle gives it a displacement $d\vec{r}$, then the work done by the force on the particle is defined as

$$dW = \vec{F} \cdot d\vec{r} = |\vec{F}||d\vec{r}|\cos\theta$$



Fig(1)

Since only the component of $\vec{F}$ in the direction of $d\vec{r}$, is effective in producing the motion.

The total work done by a force field $\vec{F}$ in moving the particle from point $P_1$ to point $P_2$ along the curve c of Fig(1) is given by the line integral

$$W = \int_c \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r}$$

where $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of $P_1$ and $P_2$ respectively. This integration is to be performed along the path of the particle.

## Kinetic Energy

In physics, a moving particle is said to have more energy than an identical particle at rest. Quantitatively the energy of moving particle is defined by

$$K(v) = \frac{1}{2}mv^2 = \frac{1}{2}m\,\vec{v}\cdot\vec{v}$$

and is called the kinetic energy of the particle.

The kinetic energy (K.E) of a system of particle is the sum of the kinetic energies of all its constituent particles i.e,

$$K = \sum_i \frac{1}{2} m v_i^2$$

The K.E of a particle or system of particles can increase or decrease or remain constant as time passes.

If no force is applied on the particle, its velocity $v$ remains constant and the K.E remains the same. So a force is necessary to change the kinetic energy of a particle. If the resultant force acting on a particle is perpendicular to its velocity, the kinetic energy does not change. So K.E changes only when speed changes and that happens only when the resultant force has a tangential ~~component~~ component.

From the definition of K.E

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v^2\right) = mv\frac{dv}{dt} = v\frac{d}{dt}(mv) = vF_t$$

where $F_t$ is the resultant tangential force. If the resultant force $\vec{F}$ makes an angle $\theta$ with velocity,

$$F_t = F\cos\theta \text{ and } \frac{dK}{dt} = F\cos\theta\, v = \vec{F}.\vec{v} = \vec{F}.\frac{d\vec{r}}{dt}$$

$$\therefore dK = \vec{F}.d\vec{r} \quad \text{——} (1)$$

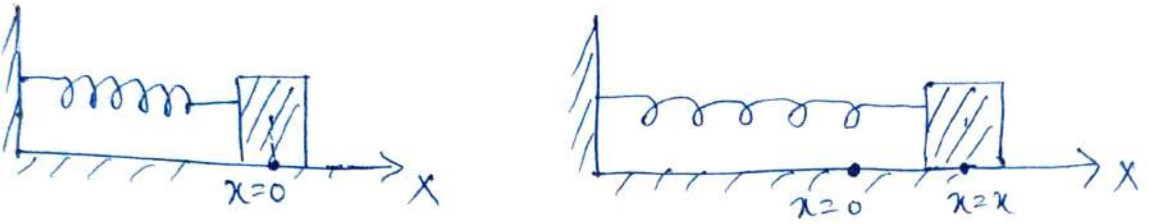
Now we have already got the total work done on the particle by a force $\vec{F}$ acting on it during a finite displacement is obtained by

$$W = \int \vec{F}.d\vec{r} = \int_{K_1}^{K_2} dK = K_2 - K_1 = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

Thus the work done on a particle by the resultant force is equal to the change in its kinetic energy. This is called the work-energy theorem.

## Calculation of work done

1. Calculate the work done when a spring is stretched.




When a spring is stretched slowly, the stretching force increases steadily as the spring elongates, ie the force is variable.

Consider the situation shown in the figure. One end of the spring is attached to a fixed vertical support and the other end to a block which can move on a horizontal table. Let $x=0$ denote the position of the block when the spring is in its natural length. We shall calculate the work done on the block by the spring force as the block moves from $x=0$ to $x=x$.

Suppose that the spring is stretched through a distance $x$ by applying the force on the block. The spring will also exert a restoring force $F$ on the block by (with elastic limit)

$$F = -kx \quad (\text{Hooke's law})$$

where $k$ is the spring constant. The $-$ve sign indicates that the restoring force is always opposite to the distance $x$.

$\therefore$ The total work done as the block is displaced from $x=0$ to $x=x$ is

$$W_1 = \int_{x=0}^{x} \vec{F}.d\vec{r} = -\int_{x=0}^{x} kx\,dx = -\frac{1}{2}kx^2$$

$$\therefore W_1 = -\frac{1}{2}kx^2$$

On the return journey the block moves from $x = x$ to $x = 0$ and the work done during return journey is given by

$$W_2 = \int_{x=x}^{x=0} \vec{F}.d\vec{r} = - \int_{x=x}^{x=0} kx\, dx = + \frac{1}{2} kx^2$$

Thus we see that the work done during the return journey is negative of the work done during the onward journey. Therefore the net work done $W = W_1 + W = -\frac{1}{2}kx^2 + \frac{1}{2}kx^2 = 0$ by the spring force in a round trip.

## Potential Energy

The second form of energy is the potential energy. A body is capable to do work by virtue of its position, configuration or state of strain. This capability of the body is called its potential energy. For example the water at the top of waterfall can rotate a turbine when falling on it. The water has this capability by virtue of its position (at a certain height). In this example the water has the gravitational potential energy.

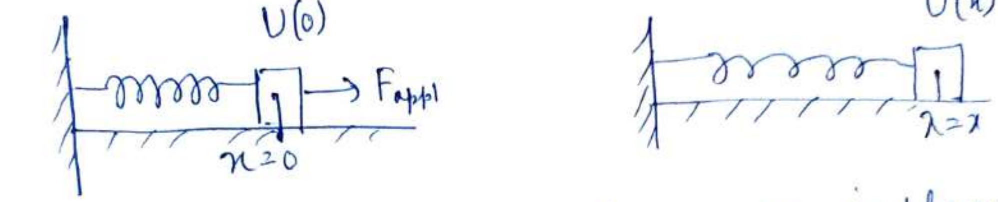
Thus the potential energy of a body or a system of bodies is a form of stored energy which can be fully recovered and converted into kinetic energy. The potential energy is denoted either by V or U.

When a body is taken from one position to another position against an intrensic conservative force (such as elastic force, gravitational force etc.), then the work done by the intrensic force in this process is stored as potential energy in the body. Thus the difference in potential energy of a body at two positions is defined as the negative of the total work done by the intrensic force in moving the body from one position to other ie, $U_f - U_i = -W = -\int_i^f \vec{F}.d\vec{r}$.

Another type of potential energy is elastic potential energy, which is associated with the state of compression or extension of an elastic (spring like) object.

If you compress or extend a spring, you do work to change the relative locations of the coils within the spring. The result of this work done is an increase in elastic potential energy of spring.

## Elastic Potential Energy (One dimensional system)




Let us consider a block of mass m is placed on the horizontal table (frictionless). One end is attached to the wall and the other end to the body. Let $x=0$ be the position of the block when the spring is in its natural length. Now give a displacement to the block from $x=0$ to $x=x$ by applying external force on the body. As a result spring will exert a force on the body which is proportional to the displacement,

$$F = -kx \quad , \quad k \text{ spring constant}$$

Let $U(0)$ be the elastic potential energy of the block at $x=0$ and $U(x)$ be the elastic potential energy at $x=x$ ie when the spring is stretched by a distance $x$.

Then the difference in potential energy

$$U(x) - U(0) = -W = -\int_{x=0}^{x} \vec{F}\cdot dx = -\int_{x=0}^{x=x} -kx\,dx = \frac{1}{2}kx^2$$

Let us assume that at $x=0$, potential energy $U(0)=0$.

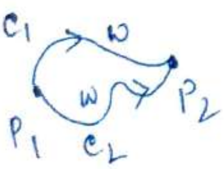
$\therefore U(x) = \frac{1}{2}kx^2 \Rightarrow$ This the potential energy of the spring when it is stretched through $x$.

# Conservative and Non-conservative forces

We divide the forces in two categories a) conservative forces b) nonconservative forces.

a) Conservative force : If the total work done by a force in moving a particle around any closed path is zero, then the force field is said to be conservative ie,
$W = \oint_C \vec{F}\cdot d\vec{r} = 0$ where $C$ is the simple closed curve.

Conservative force can also be defined as follows :
If the total work done by a force in moving a particle from one point to another point is independent of the path joining the two points, then the force field is said to be conservative.



$W = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r} \Rightarrow$ is independent of the path joining the two points $P_1$ and $P_2$ ie in all paths the work done is same.

The force of gravity, Coulomb force and spring force are conservative forces as the work done by these forces in a round trip is zero.

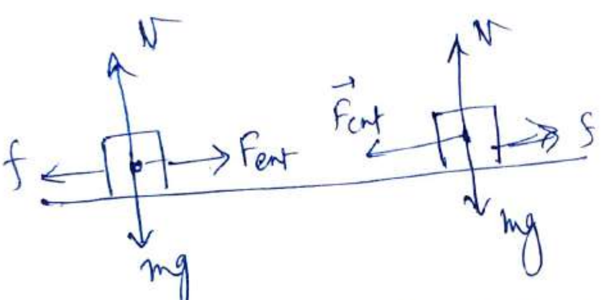
For example When we throw a ball in upward direction against gravity, the ball reaches a certain height and its velocity at that height becomes zero so that its K.E. also becomes zero. Then it returns to our hand under the gravity with the same kinetic energy with which it was thrown (provided the air resistance is assumed to be zero). Thus the change in K.E is zero. Since the change in K.E. is zero then from work energy theorem ($W = K_2 - K_1 = 0$) the total work done in a round trip is also zero. Thus the force of gravity is conservative.

## Non conservative force

A force acting on a particle is non-conservative if the particle, after going through a complete round trip, returns to its initial position with changed K.E.

For example the force of friction is nonconservative because the work done by the friction is not zero in a round trip.

Let us consider the following figure




Suppose a block of mass $m$ rests on a rough horizontal table. It is dragged horizontally towards right through a distance $l$ and then back to its initial position. Let $\mu$ be the coefficient of friction between the block and the table. Let us calculate the work done by friction during the round trip. The normal force between the table and the block is $N = mg$ and hence the force of friction is $f = \mu N = mg\mu$.

When the block moves towards right, friction on it is towards left and the work done by friction $(-\mu mgl)$ since the directions of displacement $l$ and the friction force are opposite. When the block moves towards left, the friction on it is towards right and the work done again is $(-\mu mgl)$ (because the direction of displacement and friction is opposite).

Hence the total work done by the force of friction in a round trip is $(-2\mu mgl)$ which is nonzero. Thus friction force is nonconservative.

## Definition of potential energy and conservation of mechanical energy

We define the change in potential energy of a system corresponding to a conservative internal force as

$$U_f - U_i = -W = -\int_i^f \vec{F}\cdot d\vec{r}$$

where 'W' is the work done by the internal force on the system as the system passes from the initial configuration 'i' to the final configuration 'f'.

Suppose only conservative internal forces operate between the parts of the system and the potential energy U is defined corresponding to these forces. Then we have

$$U_f - U_i = -W = -(K_f - K_i)$$

$$\text{or, } U_i + K_i = U_f + K_f \quad \text{——} (1)$$

The sum of the K.E and the P.E. is called the total mechanical energy. We see from eqn. (1) that total mechanical energy of a system remains constant if the internal forces are conservative and external forces do no work. This is called principle of conservation of energy. This implies that in a conservative force field the total energy is constant.

The total mechanical energy (K+U) is not constant if nonconservative forces such as friction act between the parts of the system. We cannot apply the principle of conservation of energy in presence of conservative forces.

But the work-energy theorem is still valid even in presence of nonconservative forces.

## Conservative force as negative gradient of potential energy

A force field $\vec{F}$ is conservative if and only if there exists a continuously differentiable scalar field V such that $\vec{F} = -\vec{\nabla} V$ or, equivalently, if and only if

$$\vec{\nabla} \times \vec{F} = \text{curl of } \vec{F} = 0 \text{ identically.}$$

Proof $\vec{F} = -\vec{\nabla} V$

If a particle acted upon by conservative force $\vec{F}$ moves from a space pt. $P_0(x_0, y_0, z_0)$ described by the position vector $\vec{r}_0$, to another pt. $Q(x, y, z)$ described by position vector $\vec{r}$, then we can write

$$\int_{P_0}^{P} dU = U(P) - U(P_0) = -W = -\int_{P_0}^{P} \vec{F} \cdot d\vec{r} \quad \text{— (1)}$$

Let us consider that potential energy at initial pt. $P_0(x_0, y_0, z_0)$ is zero i.e. $U(P_0) = 0$ and denote $U(P) = U(\vec{r})$, then we can from (1) using fundamental theorem of gradient

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz = -\vec{F} \cdot d\vec{r} = -(F_x dx + F_y dy + F_z dz) \quad \text{— (2)}$$

Comparing both sides of eqn. (2) we get

$$\frac{\partial U}{\partial x} = -F_x, \quad \frac{\partial U}{\partial y} = -F_y \text{ and } \frac{\partial U}{\partial z} = -F_z$$

Now $$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} = -\underbrace{\left( \frac{\partial U}{\partial x} \hat{i} + \frac{\partial U}{\partial y} \hat{j} + \frac{\partial U}{\partial z} \hat{k} \right)}_{\vec{\nabla} U}$$

$$\therefore \boxed{\vec{F} = -\vec{\nabla} U} \text{ proved.}$$

Proof $\vec{\nabla}\times\vec{F} = 0$ ie curl of $\vec{F}$ is zero

Since $\vec{F}$ is conservative then we can write $\vec{F} = -\vec{\nabla}U$
U is the scalar field.

$$\therefore \vec{\nabla}\times\vec{F} = \vec{\nabla}\times(-\vec{\nabla}U) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\frac{\partial U}{\partial z} - \frac{\partial}{\partial z}\frac{\partial U}{\partial y}\right] - \hat{j}\left[\frac{\partial}{\partial x}\frac{\partial U}{\partial z} - \frac{\partial}{\partial z}\frac{\partial U}{\partial x}\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}\frac{\partial U}{\partial x}\right] = 0$$

$$\therefore \boxed{\vec{\nabla}\times\vec{F} = 0} \text{ proved.}$$

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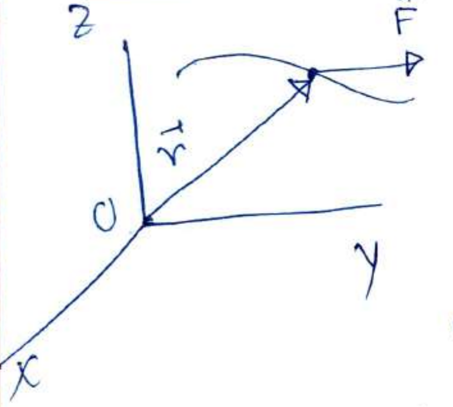
## Torque And Angular Momentum




If a particle with position vector $\vec{r}$ moves in a force field $\vec{F}$ (as shown in the figure), we define

$$\vec{\tau} = \vec{r}\times\vec{F} \quad \text{--- (1)}$$

as the torque or moment of the force $\vec{F}$ about O. The magnitude of $\vec{\tau}$ is a measure of the "turning effect" produced on the particle by the force.

We can prove that torque ($\tau$) is the time rate of change of angular moment $\vec{L}$ ($\vec{L} = \vec{r}\times\vec{p}$).

We know angular momentum $\vec{L} = m(\vec{r}\times\vec{v}) = \vec{r}\times\vec{p}$.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt}$$
$$= \vec{v}\times(m\vec{v}) + \vec{r}\times\vec{F}$$
$$= 0 + \vec{r}\times\vec{F}$$
$$= \vec{r}\times\vec{F} = \vec{\tau} \text{ --- (2)} \quad \therefore \boxed{\vec{\tau} = \frac{d\vec{L}}{dt}} \text{ proved.}$$

## Conservation of linear momentum ($\vec{p}$)

If we let $\vec{F} = 0$ in Newton's second law, we find

$$\vec{F} = \frac{d\vec{p}}{dt} = 0 \Rightarrow \vec{p} = m\vec{v} = \text{constant}$$ ie linear momentum is conserved. If the net external force acting on a particle is zero, its momentum will remain unchanged.

## Conservation of angular momentum ($\vec{L}$)

If we let $\vec{\tau} = 0$, then from the previous eqn. (2)

$$\vec{\tau} = \frac{d\vec{L}}{dt} = 0 \quad , \quad \vec{L} = m(\vec{r} \times \vec{v}) = \vec{r} \times \vec{p} = \text{constant}$$

This implies if the net external torque acting on a particle is zero, the angular momentum will remain unchanged.

This theorem is often called the principle of conservation of angular momentum.