Contour Integral (Line Integral) →

Let C be a contour $z = z(t)$, $a \le t \le b$, and $f(z) = u(x,y) + iv(x,y)$ a function which is piecewise continuous on C i.e $z(t) = x(t) + iy(t)$ then function

$$f[z(t)] = u[x(t), y(t)] + iv[x(t)+y(t)] \quad a \le t \le b$$

is piecewise continuous on the interval.
Then contour integral is defined as

$$\boxed{\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt} \quad \text{—(1)}$$

Note that since C is contour, $z'(t)$ is also piecewise continuous on the interval $a \le t \le b$ and so the existence of integral (1) is ensured.

The integrand on R.H.S of (1) is the product of the complex functions

$$u[x(t)+y(t)] + iv[x(t)+y(t)] \text{ and } x'(t)+y'(t)$$

of the variable t. Thus ~~by def~~ (1) it can be written as

$$\boxed{\int_C f(z)\,dz = \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt} \quad \text{—(2)}$$

In terms of line integrals of real-valued functions of two variable then

$$\boxed{\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy} \quad \text{—(3)}$$

Observe that expression (3) can be obtained formally by replacing f by $u+iv$ and dz by $dx+idy$ and then expanding the product.

Properties of Integrals

(1)
$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

where the contour $-C$ has the parametric representation $z = z(-t)$ $(-b \le t \le -a)$
thus $\int_{-C} f(z)\,dz = \int_{-b}^{-a} f[z(-t)]\,[-z'(-t)]\,dt$

(2) suppose that contour C consists of contours C_1 and C_2 such that the initial point of C_2 being terminal point of C_1 then

$$\int_C f(z)\,dz = \int_{(C_1+C_2)} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

(3)
$$\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$$

for any complex constant z_0.

(4)
$$\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

(5) If $|f(z)| \le M$ then

$$\left|\int_C f(z)\,dz\right| \le ML$$

where L is length of the arc.

Cauchy's Theorem – If function $f(z)$ be analytic and single valued inside and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0$$

OR.

If a function f is analytic throughout a simply connected domain D, then

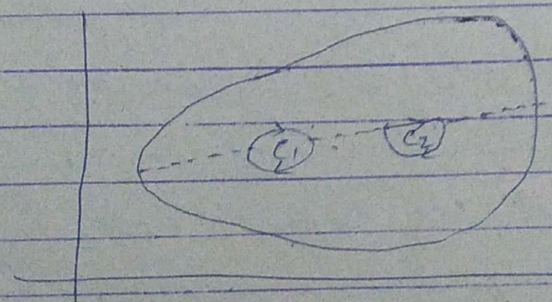
(1) $$\int f(z)\,dz = 0$$

For every simple closed contour C lying in D.

The Cauchy–Goursat theorem can also be modified in the following way, involving the boundary B of a multiply connected domain.

Theorem

Let C be a simple closed contour and C_j $(j = 1, 2 \cdots n)$ be a finite number of simple closed contours inside C such that the regions interior to each C_j have no points in common. Let R be the closed region consisting of all points within and on C except for the point interior to C_j



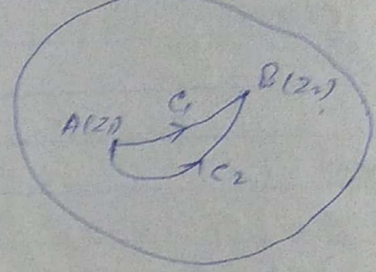
Let B denote the entire oriented boundary of R consisting of C and all the contours C_j described in a direction such that the interior points of R lie to the left of B. Then, if f is analytic throughout R

$$\int f(z)\,dz = 0.$$

Deduction from Cauchy Theorem

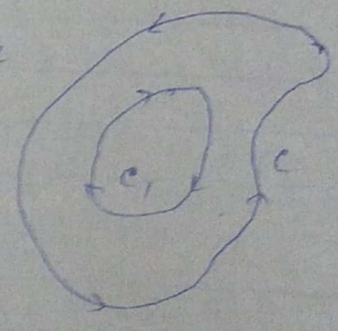
① Suppose $f(z)$ is analytic in a simply connected domain D. Then the integral along any rectifiable curve in D joining any two given points of D is the same i.e. it does not depend upon the curve joining the two points.

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$



② Let a closed contour C contain another closed contour C_1. Let $f(z)$ be analytic at every point lying in the ring shaped domain bounded by C and C_1 then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$$



③ If the contour C contains non-intersecting contours $C_1, C_2 \ldots C_n$ then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

Cauchy Integral Formula-

Let f be analytic everywhere within and on a simple closed contour C taken in the positive sense. If z_0 is any point interior C then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z-z_0} \quad \text{——} \quad (A)$$

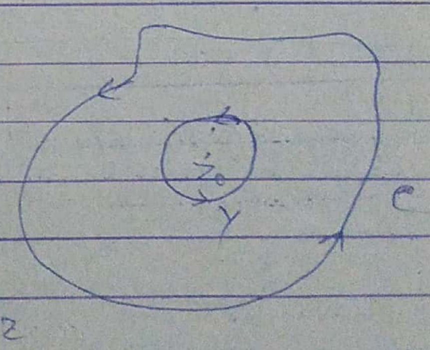
Formula (A) is called cauchy integral formula. It says that if function $f(z)$ is analytic within and on a closed contour C, then the values of f interior to C are completely determined by the values of f on C.

Proof- Suppose $f(z)$ is analytic within and on a closed contour C and z_0 is an interior point of C.

To prove that $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z-z_0}$

Describe a circle γ about the centre $z=z_0$ of small radius r s.t. this circle $|z-z_0| = r$ does not intersect the curve C. The function $\frac{f(z)}{z-z_0}$ is analytic in the annulus bounded by C and γ. Hence by deduction of cauchy theorem

$$\int_C \frac{f(z)\,dz}{z-z_0} = \int_\gamma \frac{f(z)\,dz}{z-z_0} \quad \text{——} \quad (1)$$



or $$\int_C \frac{f(z)\,dz}{z-z_0} = \int \frac{f(z)-f(z_0)}{z-z_0}\,dz + \int \frac{f(z_0)}{z-z_0}\,dz \quad \text{——} \quad (2)$$

Since $f(z)$ is analytic within C so it is continuous at $z=z_0$ so that given $\epsilon > 0 \ \exists\ \delta > 0$ s.t.

Since r is at our choice and so we can take $r < \delta$ so that ④ is satisfied $\forall z$ on the circle γ. For any point z on γ, $z - z_0 = re^{i\theta}$

$$\int \frac{f(z_0)}{z-z_0}\,dz = \int_0^{2\pi} \frac{f(z_0)\, re^{i\theta}\, i\,d\theta}{re^{i\theta}}$$
$$= 2\pi i\, f(z_0)$$

Hence by ②

$$\left| \int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) \right| = \left| \int_C \frac{f(z)-f(z_0)}{z-z_0}\,dz \right|$$

$$\leq \int_\gamma \frac{|f(z)-f(z_0)|}{|z-z_0|} \cdot |dz| < \frac{\epsilon}{r} \int_\gamma |dz|$$
$$= \frac{\epsilon}{r} \cdot 2\pi r$$

$$\text{or} \left| \int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) \right| < 2\pi\epsilon$$

since ϵ is arbitrary and so making $\epsilon \to 0$ we get

$$\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) = 0$$

$$\text{or } f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z-z_0}\,dz \qquad \text{— ⑤}$$

Remarks: ① $|a-b| < \epsilon \Rightarrow a-b=0$

② $\int_\gamma |dz|$ = circumference of circle γ
= $2\pi \cdot$ radius

③ from eqn ① and ⑤

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\,dz$$