Equivalence class: Let R be an equivalence relation on set A. Then set of all elements of R that are related to x is called equivalence class of x under R i.e,

$$[x] = \{ y : y \in A \text{ and } (x, y) \in R \}.$$

Note: $A/R = \{ [x] : x \in A \} \rightarrow$ Set of all equivalence classes of elements of A under eq. relation R is called quotient set of A by R

Ex Find distinct equivalence classes of R where $R = \{ (x, y) : x, y \in \mathbb{Z},\ xRy \Leftrightarrow x \equiv y \pmod 3 \}$.

Solution We have already proved that R is an equivalence relation.

$$\begin{aligned}
\text{Let } [a] &= \{ x \in \mathbb{Z} : xRa \}. \\
&= \{ x \in \mathbb{Z} : x - a \text{ is divisible by } 3 \}. \\
&= \{ x \in \mathbb{Z} : x - a = 3k, \text{ for some integer } k \}. \\
&= \{ x \in \mathbb{Z} : x = a + 3k \text{ for } k \in \mathbb{Z} \}.
\end{aligned}$$

Here $[a]$ is equivalence class generated by $a \in \mathbb{Z}$

$[0] = \{x \in \mathbb{Z} : x = 3k + 0 \quad k \in \mathbb{Z}\}$
$= \{\cdots -6, -3, 0, 3, 6, \cdots\}$.

$[1] = \{x \in \mathbb{Z} : x = 3k + 1, \; k \in \mathbb{Z}\}$
$= \{\cdots -5, -2, 1, 4, 7, \cdots\}$.

$[2] = \{x \in \mathbb{Z} : x = 3k + 2 ; k \in \mathbb{Z}\}$
$= \{\cdots -4, -1, 2, 5, 8 \cdots\}$

$[3] = \{x \in \mathbb{Z} : x = 3k + 3 ; k \in \mathbb{Z}\}$
$= \{x \in \mathbb{Z} : x = 3(k+1), \; k+1 \in \mathbb{Z}\}$
$= [0]$

Hence, $x \equiv y \pmod 3$ has three distinct equivalence classes, $[0], [1]$ & $[2]$.

Ex Define R on $\mathbb{Z}^+$ by $(a,b) R (c,d) \iff a+d = b+c$.
(1) Show that R is an equivalence
(2) Find $[(2,5)]$ ie, equivalence class of $(2,5)$
(3) Find $[(1,3)]$

Soln (i) $(a,a) R (a,a)$ as $a+a = a+a \;\forall a \in \mathbb{Z}^+$
$\therefore$ Reflexive

(ii) $(a,b) R (c,d) \Rightarrow a+d = b+c$
$\Rightarrow c+b = a+d \quad \Rightarrow (c,d) R (a,b)$
$\therefore$ Symmetric

(iii) $(a,b) R (c,d)$ & $(c,d) R (e,f) \Rightarrow a+d = b+c$
& $c+f = d+e$

$\Rightarrow a + \cancel{d} + \cancel{c} + f = b + \cancel{c} + \cancel{d} + e$
$\Rightarrow a+f = b+e \Rightarrow (a,b) R (e,f) \therefore$ Transitive
Hence, R is an equivalence relation

(2) $[(2,5)] = \{(a,b) : (a,b) R (2,5),\ a,b \in \mathbb{Z}^+\}$

$(a,b) R (2,5) \Leftrightarrow a+5 = b+2$

$\Rightarrow [(2,5)] = \{(1,4), (2,5), (3,6), (4,7), \ldots\}$

(3) $[(1,3)] = \{(a,b) : (a,b) R (1,3),\ a,b \in \mathbb{Z}^+\}$

$(1,3) R (a,b) \Leftrightarrow 1+b = 3+a$

or, $a+3 = b+1$

$\Rightarrow [(1,3)] = \{(1,3), (2,4), (3,5), (4,6), \ldots\}$

Ex. Let $A = \{0,1,2,3,4\}$ and $R = \{(0,0),(0,4),(1,1),(1,3),(2,2),(3,1),(3,3),(4,0),(4,4)\}$ is an equivalance relation on A. Find distinct equivalence classes of R

Soln ~~$[0] = \{(0,0),(0,4)\}$~~ $[0]$ = {set of ordered pair of elements related to '0'}

~~$[1] = \{(1,1),(1,3),(3,1)\}$~~

~~$[2] = \{(2,2)\}$~~

$[0] = \{0,4\} = [4]$

$[1] = \{1,3\} = [3]$

$[2] = \{2\}$.

Ex Let R be relation congruence modulo3. Which of the following equivalence classes are equal?

$[7], [-4], [-6], [17], [4], [27], [19]$

Soln $x \equiv y \pmod 3 \Rightarrow x-y = 3k \quad k \in \mathbb{Z}$

$7 \equiv x \pmod 3 \Rightarrow 7 = 3k+x \Rightarrow x=1,\ k=1$

$4 \equiv x \pmod 3 \Rightarrow 4 = 3k+x \Rightarrow x=1,\ k=1$

$\Rightarrow [4] = [7]$

$19 \equiv x \pmod 3 \Rightarrow 19 = 3k+x \Rightarrow x=1 \quad k=6$

$\therefore [4] = [7] = [19]$

$-4 \equiv x \pmod 3 \Rightarrow -4 = 3k + x \Rightarrow x = 2 \quad k = -2$
$17 \equiv x \pmod 3 \Rightarrow 17 = 3k + x \Rightarrow x = 2 \quad k = 5$
$\therefore [-4] = [17]$

$-6 \equiv 3 \pmod 3 \Rightarrow x = 0$
$27 \equiv 3 \pmod 3 \Rightarrow x = 0 \Rightarrow [-6] = [27]$ //

<u>Theorem</u>: Let A be a non-empty and R be an equivalence relation defined on A. Let $a, b \in A$ be aribitary. Then
(1) $a \in [a]$ (2) $b \in [a] \Rightarrow [b] = [a]$
(3) $[a] = [b] \Leftrightarrow (a,b) \in R$ (4) Either $[a] = [b]$ or $[a] \cap [b] = \phi$

<u>Sol</u>: (1) R is an eq. rel. $\Rightarrow R$ is ~~sy~~ reflexive
$\Rightarrow aRa \; \forall a \in A \Rightarrow a \in [a]$ //

(2) $b \in [a] \Rightarrow (b,a) \in R \Rightarrow (a,b) \in R$ ($\because R$ is symmetric)
Let $x \in [b] \Rightarrow xRb$ and $bRa \Rightarrow xRa$ (R is transitive)
$\Rightarrow x \in [a]$
$\Rightarrow [b] \subseteq [a]$ —(i)
Let $x \in [a] \Rightarrow xRa$ & $aRb \Rightarrow xRb \Rightarrow x \in [b]$
(transitivity)
$\Rightarrow [a] \subseteq [b]$ —(ii)
From (i) & (ii) $[a] = [b]$ //

(3) $[a] = [b]$. Let $x \in [a] = [b]$
$\Rightarrow xRa$ & $xRb \Rightarrow xRa$ & bRx symmetric
$\Rightarrow bRa$ (transitivity)
$\Rightarrow (a,b) \in R$ //

Conversely: Let $(a,b) \in R$
Let $x \in [a] \Rightarrow xRa$ & $aRb \Rightarrow xRb$ (transitivity)
$\Rightarrow x \in [b] \Rightarrow [a] \subseteq [b]$ —(I)
Let $x \in [b] \Rightarrow xRb$ & $bRa \Rightarrow xRa \Rightarrow x \in [a]$
$\Rightarrow [b] \subseteq [a]$ —(II) From (I) & (II) $[a] = [b]$ //

(4) Assume $[a] \cap [b] \neq \phi$
$\Rightarrow \exists\, x \in [a] \cap [b]$.
$\Rightarrow x \in [a]$ & $x \in [b]$
$\Rightarrow xRa$ & xRb
$\Rightarrow aRx$ & xRb $\Rightarrow aRb$. $\Rightarrow [a]=[b]$ (from (3))
Hence, either $[a]=[b]$ or $[a]\cap[b]=\phi$ //.

Note: From above theorem it is clear that either 2 eq. classes are identical or completely disjoint.

Partition of Set: A partition of a set A is set of non-empty subsets of A denoted by $\{A_1, A_2, \ldots A_n\}$ such that
(i) $\bigcup_{i=1}^{n} A_i = A$ (ii) $A_i \cap A_j = \phi$ $i \neq j$

Ex: $A = \{0,1,2,3,4\}$ then $A_1 = \{0,1\}$, $A_2 = \{2,3\}$, $A_3 = \{4\}$ is a partition of A as $\bigcup_{i=1}^{3} A_i = A$ & $A_i \cap A_j = \phi$ & $A_i \neq \phi$.

Thm Let R be an equivalence relation on set A. Then the distinct equivalence classes of R form a partition of A

Pf: Let $A_1, A_2, \ldots A_n$ be distinct equivalence classes of R.
T.P.T (1) $A = \bigcup_{i=1}^{n} A_i$ (2) $A_i \cap A_j = \phi$ $i \neq j$

(1) R is reflexive $\Rightarrow xRx\ \forall x \in A$
$\Rightarrow x \in [A_i]$ [$\because A_i$'s are equivalence classes formed from elements of A].
$\Rightarrow x \in \bigcup_{i=1}^{n} A_i$ $\Rightarrow A \subseteq \bigcup_{i=1}^{n} A_i$ — (i)
Let $x \in \bigcup_{i=1}^{n} A_i$ $\Rightarrow x \in A_j$ for some j, $1 \le j \le n$
A_j is an eq. class of R & hence, $A_j \subseteq A$, $1 \le j \le n$
$\Rightarrow \bigcup_{i=1}^{n} A_i \subseteq A$ — (ii) From (i) & (ii) $A = \bigcup_{i=1}^{n} A_i$ //

(2) Let A_i and A_j be distinct eq. classes of R . i.e, $A_i \neq A_j$

$\Rightarrow$ Either $A_i = A_j$ or $A_i \cap A_j = \phi$. {from Thm (4) above}

But $A_i \neq A_j$ $\Rightarrow$ $A_i \cap A_j = \phi$, $i \neq j$ //

Ex Let $A = \{a, b, c, d, e\}$ and $R = \{(a,b), (a,a), (b,a), (b,b), (c,c), (d,d), (d,e), (e,e), (e,d)\}$ be an eq rel on A . Determine partition of (1) R (2) R^{-1}

Soln (1) $[a] = \{a, b\} = [b]$

$[c] = \{c\}$ $[d] = \{e, d\} = [e]$

$\therefore$ Partition of $R = \{\{a, b\}, \{c\}, \{e, d\}\}$.

(2) Partition of R = partition of R^{-1} (why ??)

Ex Let R be an eq rel on set $A = \{1, 2, 3, 4\}$ defined by partitions $P = \{\{1, 4\}, \{2, 3\}\}$. Determine the elements of eq rel and also find eq classes of R.

Soln $R = \{(1,1), (4,4), (1,4), (4,1), (2,2), (3,3), (2,3), (3,2)\}$

$[1] = \{1, 4\}$ $[2] = \{2, 3\} = [3]$ //
$[4]$