(a) $\sim(p \to (q \wedge r))$

$\Leftrightarrow \sim(\sim p \vee (q \wedge r)) \Leftrightarrow (p \wedge (\sim q \vee \sim r))$ (De Morgan law)

$\Leftrightarrow (p \wedge \sim q) \vee (p \wedge \sim r)$ (distributive prop) — (1).

$(p \wedge \sim q) \Leftrightarrow (\underbrace{p \wedge \sim q}_{\alpha}) \wedge (r \vee \sim r)$ (∵ r is missing)

$\Leftrightarrow \cancel{p}(\alpha \wedge r) \vee (\alpha \wedge \sim r)$

$\Leftrightarrow (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r)$ — (2)

$(p \wedge \sim r) \Leftrightarrow (\underbrace{p \wedge \sim r}_{\alpha}) \wedge (q \vee \sim q)$ (∵ q is missing)

$\Leftrightarrow \cancel{p}(\alpha \wedge q) \vee (\alpha \wedge \sim q)$

$\Leftrightarrow (p \wedge \sim r \wedge q) \vee (p \wedge \sim r \wedge \sim q)$

$\Leftrightarrow (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge \sim r)$ — (2)

From (1), (2) & (3), we get

$\sim(p \to (q \wedge r)) \Leftrightarrow (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (p \wedge q \wedge \sim r) \vee \cancel{(p \wedge \sim q \wedge \sim r)}$ (repeated)

$\Leftrightarrow (p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge \sim r)$

— Ans

(c) $(p \wedge (\sim(q \wedge r))) \vee (p \to q)$

$\Leftrightarrow [p \wedge (\sim(q \wedge r))] \vee (\sim p \vee q)$

$\Leftrightarrow \{[p \wedge (\sim(q \wedge r))] \vee \sim p\} \vee q$ (Associative law)

$\Leftrightarrow [(\underbrace{p \vee \sim p}_{T}) \wedge \{\sim(q \wedge r)) \vee \sim p\}] \vee q$

$\Leftrightarrow \{\sim(q \wedge r) \vee \sim p\} \vee q$ (∵ $T \wedge A \Leftrightarrow A$)

$\Leftrightarrow \sim q \vee \sim r \vee \sim p \vee q$

$\Leftrightarrow (\underbrace{q \vee \sim q}_{T}) \vee \sim r \vee \sim p$ (associative law)

$\Leftrightarrow T \vee \sim r \vee \sim p$

$\Leftrightarrow T \qquad (\because T \vee A \Leftrightarrow T)$

$\Leftrightarrow (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r)$
$\vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$
$\vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r)$ — Ans

b) $(\sim p \to r) \wedge (p \leftrightarrow q)$
$\Leftrightarrow (p \vee r) \wedge (p \to q) \wedge (q \to p)$
$\Leftrightarrow (p \vee r) \wedge (\sim p \vee q) \wedge (\sim q \vee p)$
(Do it later using different method which will be much simpler).

Ex obtain PDNF of $p \vee (\sim p \wedge \sim q \wedge r)$
$p \vee (\sim p \wedge \sim q \wedge r)$

$p \Leftrightarrow p \wedge (q \vee \sim q) \Leftrightarrow (p \wedge q) \vee (p \wedge \sim q)$ — ① (inserting q)

$(p \wedge q) \Leftrightarrow \underbrace{(p \wedge q)}_{\alpha} \wedge (r \vee \sim r) \Leftrightarrow (\alpha \wedge r) \vee (\alpha \wedge \sim r)$

$\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r)$ — ②

$(p \wedge \sim q) \Leftrightarrow \underbrace{(p \wedge \sim q)}_{\alpha} \wedge (r \vee \sim r) \Leftrightarrow (\alpha \wedge r) \vee (\alpha \wedge \sim r)$

$\Leftrightarrow (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r)$ — ③

$\therefore p \Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r)$
$\vee (p \wedge q \wedge \sim r)$ repeated

$\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r)$

$\therefore p \vee (\sim p \wedge \sim q \wedge r)$
$\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r)$ — Ans

Ex(d)

$\underbrace{(p \vee q)}_{\alpha} \wedge (\sim p \wedge \sim q)$

$\Leftrightarrow \alpha \wedge (\sim p) \wedge (\sim q)$

$\Leftrightarrow (\alpha \wedge (\sim p)) \wedge (\sim q)$

$\Leftrightarrow [(p \vee q) \wedge (\sim p)] \wedge (\sim q)$

$\Leftrightarrow [\underbrace{(p \wedge \sim p)}_{F} \vee (q \wedge \sim p)] \wedge (\sim q)$

$\Leftrightarrow (\sim p \wedge q) \wedge (\sim q)$ $\quad (\because F \vee A \Leftrightarrow A.)$

$\Leftrightarrow \sim p \wedge \underbrace{(q \wedge \sim q)}_{F}$

$\Leftrightarrow F$ $\quad (\because F \wedge A \Leftrightarrow F$

$\Rightarrow$ PDNF does not exists as it is a contradiction.

Ex Write in PDNF $(\sim p \wedge q) \vee (p \wedge \sim r)$

$\Leftrightarrow [\underbrace{(\sim p \wedge q)}_{\alpha} \wedge (r \vee \sim r)] \vee [(p \wedge \sim r) \wedge (q \vee \sim q)]$

$\Leftrightarrow (\underbrace{\sim p \wedge q}_{\alpha} \wedge r) \vee (\underbrace{\sim p \wedge q}_{\alpha} \wedge \sim r) \vee (p \wedge \sim r \wedge q) \vee (p \wedge \sim r \wedge \sim q)$

$\Leftrightarrow (\sim p \wedge q \wedge r) \vee (p \wedge \sim r \wedge q) \vee (\sim p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge \sim r)$ Ans

Principal Conjunctive Normal form :

maxterms: Elementary sum in which variable and its negation do not occur simultaneously

eg maxterms of p & q : $p \vee \sim q$, $\sim p \vee q$, $p \vee q$, $\sim p \vee \sim q$.

n-variables $2^n \to$ maxterms.

p	q	$p \vee q$	$\sim p \vee q$	$p \vee \sim q$	$\sim p \vee \sim q$
T	T	T	T	T	F
T	F	T	F	T	T
F	T	T	T	F	T
F	F	F	T	T	T

<u>Note</u>: Each maxterm has truth value F for exactly one combination of truth values. This makes each maxterm unique
PCNF = Product of maxterms (hence PCNF is also unique)

<u>Using Truth Table</u>:
(a) $\forall$ truth value F, select a maxterm which has value F for the same combination of truth values of statement variables.
(b) Take product of maxterms in step a

<u>Ex</u> Obtain PCNF of
(a) $\sim(p \to (q \wedge r))$ (b) $(\sim p \to r) \wedge (p \leftrightarrow q)$
(c) $(p \wedge \sim(q \wedge r)) \vee (p \to q)$ (d) $(p \vee q) \wedge (\sim p \vee \sim q)$

(a) From truth table
1^{st} row $\to$ $\sim p \vee \sim q \vee \sim r$ 5^{th} row $\to$ $p \vee \sim q \vee \sim r$
6^{th} row $\to$ $p \vee \sim q \vee r$ 7^{th} row $\to$ $p \vee q \vee \sim r$
8^{th} row $\to$ $p \vee q \vee r$
$\therefore \sim(p \to (q \wedge r)) \Leftrightarrow (p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee \sim r)$
$\wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee \sim q \vee \sim r)$

b) $(\sim p \to r) \wedge (p \leftrightarrow q) \Leftrightarrow (\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (p \vee \sim q \vee \sim r)$
$\wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee r)$

$\Leftrightarrow (p\vee q\vee r)\wedge(\sim p\vee q\vee r)\wedge(p\vee\sim q\vee r)\wedge(\sim p\vee q\vee\sim r)$
$\wedge(p\vee\sim q\vee\sim r)$

(c) $(p\wedge\sim(q\wedge r))\vee(p\rightarrow q)\rightarrow$ Tautology hence PCNF does not exist

(d) $(p\vee q)\wedge(\sim p\wedge\sim q)\Leftrightarrow$ Contradiction
$\Leftrightarrow (p\vee q)\wedge(\sim p\vee q)\wedge(p\vee\sim q)\wedge(\sim p\vee\sim q)$

<u>Without using truth table</u>:

(1) Remove all $\rightarrow$ & $\rightleftharpoons$ by $\sim, \vee, \wedge$ only
(2) Eliminate $\sim$ before sums & products using De Morgan's law
(3) Apply distributive property.
(4) Drop terms which are tautology (ie, $p\vee\sim p$)
(5) Introduce the missing variable in elementary sum by taking its sum with contradiction ie, $(p\vee q)\Leftrightarrow(p\vee q)\vee(r\wedge\sim r)$
(6) Repeat step 5 till all elementary sums are reduced to product of maxterms
(7) Delete identical maxterms.

<u>Ex</u> Find PCNF of

(a) $p\wedge(p\vee\sim q\vee r)$

Ans: Consider

$p\Leftrightarrow p\vee(q\wedge\sim q)$ (introducing q)
$\Leftrightarrow (p\vee q)\wedge(p\vee\sim q)$
$\Leftrightarrow [(p\vee q)\vee(r\wedge\sim r)]\wedge[(p\vee\sim q)\vee(r\wedge\sim r)]$ (introducing r)
$\Leftrightarrow (p\vee q\vee r)\wedge(p\vee q\vee\sim r)\wedge(p\vee\sim q\vee r)\wedge(p\vee\sim q\vee\sim r)$
$\Leftrightarrow (p\vee q\vee r)\wedge(p\vee\sim q\vee r)\wedge(p\vee q\vee\sim r)\wedge(p\vee\sim q\vee\sim r)$

therefore, p^(pv~qvr)= (as (pv~qvr) is already in expression for p)

(b) $(p \wedge \sim(q \wedge r)) \vee (p \to q)$

$\Leftrightarrow [p \wedge (\sim q \vee \sim r)] \vee (\underbrace{\sim p \vee q}_{\alpha})$

$\Leftrightarrow (p \vee \alpha) \wedge [(\sim q \vee \sim r) \vee \alpha]$

$\Leftrightarrow (\underbrace{p \vee \sim p}_{T} \vee q) \wedge [\sim q \vee \sim r \vee \sim p \vee q]$ T

$\Leftrightarrow T \wedge T \Leftrightarrow T$

$\Rightarrow$ PCNF does not exist.

(c) $(\sim p \to r) \wedge (p \leftrightarrow q) \Leftrightarrow (\sim p \to r) \wedge (p \to q) \wedge (q \to p)$

$\Leftrightarrow (p \vee r) \wedge (\sim p \vee q) \wedge (\sim q \vee p)$ — (1)

$(p \vee r) \Leftrightarrow (\underbrace{p \vee r}_{\alpha}) \vee (q \wedge \sim q)$ (introducing q)

$\Leftrightarrow (\alpha \vee q) \wedge (\alpha \vee \sim q)$

$\Leftrightarrow (p \vee r \vee q) \wedge (p \vee r \vee \sim q)$ (2)

$\Leftrightarrow (p \vee q \vee r) \wedge (p \vee \sim q \vee r)$ (Commulative prop)

Similarly $(\sim p \vee q) \Leftrightarrow (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

and $(p \vee \sim q) \Leftrightarrow (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r)$ — (3)

∴ From (1), (2) and (3), we have

$(\sim p \to r) \wedge (p \leftrightarrow q) \Leftrightarrow (p \vee q \vee r) \wedge (\sim p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee \sim q \vee \sim r)$

(d) $(p \vee q) \wedge (\sim p \vee \sim q) \to$ Already in PCNF.