Neighbourhood of a point (1)

Let (X, T) be a topological space. A subset N of X is called a T-nbd of $x \in X$ iff $\exists$ a T-open set G s.t.

$$x \in G \subseteq N$$

Thus a T-nbd of x is superset of T-open set containing x.

T-open set is a T-nbd of each of its points

Example Let $X = \{1, 2, 3, 4, 5\}$ and let

$T = \{\phi, \{1\}, \{1,2\}, \{1,2,5\}, \{1,3,4\}, \{1,2,3,4\}, X\}$

The T-nbd of $3 \in X$ are

$\{1,3,4\}$ $\{1,2,3,4\}$, X, $\{1,5,3,4\}$,

and T-nbd of $5 \in X$ are $\{1,2,5\}$, X, $\{1,2,3,5\}$ and $\{1,2,4,5\}$.

In a Indiscrete topological space the T-nbd of each its point is X itself.

The collection of all T-nbds of $x \in X$ is called the nbd system at x and is denoted by $N(x)$

So far, we have used the open set as primitive notion for a topology. Also nbd can be used as primitive notion to define a topology.

Limit points, Isolated points and Adherent points

Let (X, T) be a topological space and let A is a subset of X. A point $x \in X$ is called a limit point of A iff every nbd of x contains a point of A other than x

Thus x is a limit point of A iff
$(N - \{x\}) \cap A \neq \phi$ $\forall$ T-nbd N of x

The set of all limit points of A is called the derived set of A and is denoted by $D(A)$

A point x is called an isolated point of set $A \subset X$ if x belongs A but x is not a limit point of A i.e. $\exists$ some nbd N of x s.t. N contains no point of A other than x.

A closed set which has no isolated points is said to be perfect.

A point $x \in X$ is called an adherent point of the set $A \subset X$ (or contact point) iff every nbd of x contains a point of A.

The set of all adherent points of A is called adherence of A and is denoted by $Adh(A)$

<u>Example</u>- Let (X, T) be a topological space where
$X = \{a, b, c, d, e\}$
and $T = \{\phi, \{b\}, \{d, e\}, \{b, d, e\}, \{a, c, d, e\}, X\}$
find the limit points, derived set, isolated points and the adherent points of the set $A = \{b, c, d\}$

<u>Soln.</u> The T-nbd of a are $\{a, c, d, e\}$ & X and both contain a point of A other than x so a is a limit point of A. Since a T-nbd of b, $\{b\}$ does not contain any point of A other than b, so b is not limit point A. On the same fact the other limit points are c & e and
$D(A) = \{a, c, e\}$.

Isolated points of A are b and d since b and d belong to A and are not limit points of A.

The adherent points of A are a, b, c, d, e

CLOSURE – Let (X, T) be a topological space and $A \subset X$. The intersection of all T-closed supersets of A is called the closure of A and is denoted by $\overline{A}$. Thus

$$\overline{A} = \cap \{F : F \text{ is } T\text{-closed } \& \ F \supset A\}$$

Let A be a subset of a topological space then (i) $\overline{A}$ is the smallest closed set containing A

(ii) A is closed iff $\overline{A} = A$

(iii) $\overline{A} = A \cup D(A)$

(iv) $\overline{A} = \text{Adh } A$

Properties of closure

Let A and B are two subsets of a topological space (X, T) then

(i) $\overline{\phi} = \phi$ (ii) $A \subset \overline{A}$

(iii) $A \subset B \Rightarrow \overline{A} \subset \overline{B}$ (iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(v) $(\overline{A \cap B}) \subset \overline{A} \cap \overline{B}$ (vi) $\overline{(\overline{A})} = \overline{A}$

Ex. Let (R, T) be the usual topological space then find T-closure of the following subsets of R

(i) $A = \{\frac{1}{n} : n \in N\}$ (ii) I (iii) Q (iv) $(0, 1)$

Soln (i) Since $D(A) = \{0\} \Rightarrow \overline{A} = A \cup D(A) = \{0, 1, \frac{1}{2} \cdots\}$

(ii) Since $D(I) = \phi \Rightarrow \overline{I} = I$

(iii) Since $D(Q) = R \Rightarrow \overline{Q} = R$

(iv) Since $D(0,1) = [0, 1] \Rightarrow \overline{(0,1)} = [0, 1]$

INTERIOR And Interior point

Let (X, T) be a Topological space and let $A \subset X$. A point $x \in A$ is called an interior point of A iff A is a nbd of x i.e. iff $\exists$ an T-open set G s.t. $x \in G \subset A$

The set of all interior points of A is called its interior and is denoted by A^0 or Int.(A)

Thus $A^0 = \cup \{G : G \text{ is T-open}, G \subset A\}$

Properties of Interior

(i) $X^0 = X$. $\phi^0 = \phi$ (ii) $A \subset B \Rightarrow A^0 \subset B^0$

(iii) $(A \cap B)^0 = A^0 \cap B^0$ (iv) $A^0 \cup B^0 \subset (A \cup B)^0$

(v) $A^{00} = A^0$ (vi) A is open iff $A^0 = A$

(vii) A^0 is the largest open set contained in A.

Exterior points and Exterior

Let A be a subset of a topological space (X, T). A point $x \in X$ is called an exterior point of A iff x is an interior point of A' i.e. iff $\exists$ an T-open set G s.t. $x \in G \subset A'$

The set of all exterior points of A is called exterior of A and is denoted by ext(A)

Thus $\text{Ext}(A) = \cup \{G \in T : G \subset A'\}$

NOTE- $\text{ext}(A) = (\bar{A})'$

Properties of Exterior

(i) $\text{ext}(X) = \phi$, $\text{ext}(\phi) = X$ (ii) $\text{ext}(A) \subset A'$

(iii) $A \subset B \Rightarrow \text{ext}(B) \subset \text{ext}(A)$

(vi) $A^0 \subset \text{ext}(\text{ext}\, A)$

Frontier points and frontier of a set

Let (X, T) be a topological space. A point $x \in X$ is called a frontier (or boundary) point of $A \subset X$ iff x is neither an interior nor an exterior point of A i.e. iff $\exists$ T-open set G containing x s.t. $G \cap A \neq \phi$ and $G \cap A' \neq \phi$ or iff every nbd of x intersects A and A' both.

The set of all boundary points of A is called boundary (or frontier) of A and is denoted by $Fr(A)$ or $b(A)$

Note - $Fr(A) = Fr(A')$

Ex. Let $X = \{a, b, c, d, e\}$ and let

$$T = \{\phi, \{b\}, \{c, d\}, \{b, c, d\}, \{a, b, c, d\}, X\}$$

Find (i) Interior (ii) Exterior (iii) frontier of the set

$$A = \{a, b, d\}$$

Soln.

(i) The T-open sets contained in A be $\{b\}$

$$\therefore A^0 = \{b\}$$

(ii) $A' = \{c, e\}$

the T-open sets contained in A' be ϕ

$$\therefore ext(A) = \phi$$

(iii) $\because A^0 = \{b\}$ and $ext(A) = \phi$

It follows that

$$Fr(A) = \{a, c, d, e\}$$

NOTE — $\bar{A} = A^0 \cup Fr(A)$

and if A is open $Fr(A) = \bar{A} - A$