# Relations:

Let A and B be two non-empty sets. A relation from A to B is subset of $A \times B$ denoted R. ie,
$R \subseteq A \times B = \{(x,y) : x \in A, y \in B\}$

* If $x$ is related to $y$ by a relation R then we write $xRy$ or $(x,y) \in R$.

* If $\#(A) = m$ & $\#(B) = n$ then there can be atmost $2^{mn}$ relations from A to B.

Ex Let $A = \{2,4,6,8\}$ $B = \{1,3,5,7\}$ & $R \subseteq A \times B$.

1) $xRy$ iff $x < y$. $R = \{(2,3),(2,5),(2,7)$
$(4,5),(4,7),(6,7)\}$.

2) $xRy$ iff $x = y+1$ $R = \{(2,1),(4,1),(6,5),(8,7)\}$

(3) $xRy$ iff $x+y$ is divisible by 3.
$R = \{(2,1),(2,7),(4,5),(6,3),(8,1)$
$(8,7)\}$.

Similarly many other relations can be defined from A to B.

* These are called Binary Relations.
* If $x$ is not related to $y$ via relation R then we write $x \not R y$ or $(x,y) \notin R$.

Domain of Relation: Set of first element of every ordered pair of R. $D = \{x : x \in A, (x,y) \in R, y \in B\}$.

Range of Relation: set of 2nd element of every ordered pair of R. $R = \{y : y \in B, (x,y) \in R, x \in A\}$.

Ex I let $R = \{(2,1), (2,5), (4,3), (4,7), (6,7), (8,3)\}$ be relation from A to B. Then
Domain $R = \{2,4,6,8\}$
Range $R = \{1,5,3,7\}$.

* In other words if $R \subseteq A \times B$ then domain $(R) \subseteq A$ & range $(R) \subseteq B$.

<u>Inverse Relation</u>: If $R \subseteq A \times B$ then
$R = \{(x,y) : x \in A, y \in B$ and $xRy\}$.

Then $R^{-1}$ is relation from B to A defined by
$R^{-1} = \{(y,x) : (x,y) \in R, x \in A, y \in B\}$.

<u>Ex</u> Let R be relation in Ex I above. Then
$R^{-1} = \{(1,2), (5,2), (3,4), (7,4), (7,6), (3,8)\}$.

* Domain $(R)$ = Range $(R^{-1})$
Range $(R)$ = Domain $(R^{-1})$.

* If $A = B$ then we say R is relation on set A. and $R \subseteq A \times A$.

<u>Ex. II</u> Let R and S be two relations from A to B. Show that
(a) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ (b) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

<u>Soln</u> (a) Let $(x,y) \in (R \cap S)^{-1} \Leftrightarrow (y,x) \in R \cap S$
$\Leftrightarrow (y,x) \in R$ and $(y,x) \in S$
$\Leftrightarrow (x,y) \in R^{-1}$ and $(x,y) \in S^{-1}$
$\Leftrightarrow (x,y) \in R^{-1} \cap S^{-1}$ //

(b) Let $(x,y) \in (R\cup S)^{-1} \Leftrightarrow (y,x) \in R\cup S$

$\Leftrightarrow (y,x) \in R$ or $(y,x) \in S$

$\Leftrightarrow (x,y) \in R^{-1}$ or $(x,y) \in S^{-1}$

$\Leftrightarrow (x,y) \in R^{-1}\cup S^{-1}$ //.

**Composition of Relation:** Let $R$ be relation from $A$ to $B$ and $S$ be relation from $B$ to $C$. Then $S \circ R$ is a relation from $A$ to $C$, defined by

$$S\circ R = \{(x,z) : \exists\, y \in B \text{ s.t. } (x,y) \in R \text{ and } (y,z) \in S\}.$$
$$x\in A,\ z \in C$$

i.e, $(x,y) \in R$ & $(y,z) \in S \Leftrightarrow (x,z) \in S\circ R$. — (*)

**Theorem:** If $R^{-1}$ and $S^{-1}$ are inverse of relations $R$ and $S$ respectively, then $(S\circ R)^{-1} = R^{-1}\circ S^{-1}$.

**Pf:** Let $R \subseteq A\times B$ and $S \subseteq B\times C$. Then $S\circ R \subseteq A\times C$

Then

Let $(z,x) \in (S\circ R)^{-1}$ (i.e, $(x,z) \in S\circ R$)

$\Rightarrow (x,z) \in S\circ R$.

$\Rightarrow \exists\, y\in B$ such that $(x,y) \in R$ and $(y,z) \in S$

$\Rightarrow (y,x) \in R^{-1}$ and $(z,y) \in S^{-1}$

$\Rightarrow (z,x) \in R^{-1}\circ S^{-1}$ (see (*))

$\Rightarrow (S\circ R)^{-1} \subseteq R^{-1}\circ S^{-1}$ — (1)

Similarly prove $R^{-1}\circ S^{-1} \subseteq (S\circ R)^{-1}$ — (2)

(Just revert the steps above)

$\therefore$ From (1) & (2) $(S\circ R)^{-1} = R^{-1}\circ S^{-1}$. //

**Do it:**

**Ex** If $R$ & $S$ are relation on $A$ then prove that $R\cup S$, $R-S$ are also relations on $A$.

Relation on a set: If $A = B$ then $R \subseteq A \times A$ and we say $R$ is relation on set $A$.

Identity relation: on set $A$ is defined as
$I_A = \{(x, x) : x \in A\}$.

Clearly domain $(I_A)$ = range $(I_A)$.

Types of relations:

1. Reflexive Relation: Let $R$ be a relation on set $A$. Then $R$ is reflexive if $xRx \ \forall x \in A$
or, $(x, x) \in R \ \forall x \in R$.

e.g. Let $A = \{1, 2, 3\}$ & $R = \{(1,2), (1,1), (2,2), (3,2)\}$

Then $R$ is not reflexive as $(3,3) \notin R$.

2. Symmetric Relation: A relation $R$ on set $A$ is said to be symmetric if $aRb$ then $bRa$
or, if $(a,b) \in R$ then $(b,a) \in R$ ; $a, b \in A$

e.g. $A = \{1, 2, 3\}$ $R = \{(1,2), (1,1), (3,3), (2,1)\}$ is a symmetric relation

3. Anti-Symmetric Relation: A relation $R$ on set $A$ is said to be anti-symmetric if $aRb$ & $bRb$ then $a=b$
or, if $(a,b) \in R$ & $(b,a) \in R$ then $a = b$.

Let $A = \mathbb{N}$ and $R \to$ a divides $b$. Then if $a$ divides $b$ & $b$ divides $a$ then $a = b$.

4. **Transitive Relation:** Let $R$ be a relation on set $A$. Then $R$ is called transitive if $aRb$ and $bRc$ then $aRc$ i.e, $(a,b)\in R,\ (b,c)\in R \Rightarrow (a,c)\in R$.

Ex Let $A=\{1,2,3\}$ & $R=\{(1,2),(1,3),(2,3)\}$.
Then $R$ is transitive. $(1,2),(2,3)\in R$
$\Rightarrow (1,3)\in R$.

Ex If $A=\{1,2,3,--10\}$ and the relation $R$ is defined such that $xRy \Leftrightarrow x+2y=0$, $x,y\in A$ then find the (1) domain$(R)$ (2) range$(R)$ (3) $R^{-1}$

Soln $xRy \Leftrightarrow x-2y=0$ or, $x=2y$
$\therefore R=\{(2,1),(4,2),(6,3),(8,4),(10,5)\}$

(1) domain$(R)=\{2,4,6,8,10\}$
(2) range$(R)=\{1,2,3,4,5\}$.
(3) $R^{-1}=\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$.

Ex In above example $xRy \Leftrightarrow x+2y=10$. Find $R$, $R^{-1}$

Soln $x=2, y=4$; $x=4, y=3$; $x=6, y=2$, $x=8; y=1$
$\therefore R=\{(2,4),(4,3),(6,2),(8,1)\}$
and $R^{-1}=\{(4,2),(3,4),(2,6),(1,8)\}$.

Ex If $A=\{1,2,3,4\}$, then which of the following relations are (1) reflexive (2) symmetric (3) transitive
(1) $R=\{(1,2),(2,2),(3,4),(4,3),(2,3),(3,2),(3,3)\}$
(2) $R=\{(1,2),(2,1),(1,1),(2,2),(3,4)\}$
(3) $R=\{(1,1),(2,2),(3,3)\}$

(1) $R = \{(1,2),(2,2),(3,4),(4,3),(2,3),(3,2),(3,3)\}$

(a) Not reflexive as $(1,1),(4,4) \notin R$.

(b) Not symmetric as $(1,2) \in R$ & $(2,1) \notin R$

(c) Not transitive as $(4,3);(3,4) \in R \not\Rightarrow (4,4) \in R$

This is not anti-symmetric (why ?!)

(2) $R = \{(1,2),(2,1),(1,1),(2,2),(3,4)\}$

(a) Not reflexive: $(3,3),(4,4) \notin R$

(b) Not symmetric as $(3,4) \in R$ but $(4,3) \notin R$

(c) Transitive.

(d) Not ~~tran~~ antisymmetric as $(1,2);(2,1) \in R$ but $1 \neq 2$.

(3) $R = \{(1,1),(2,2),(3,3)\}$.

(1) Reflexive (2) Symmetric (3) transitive

(4) antisymmetric.

Ex Let $R$ be relation on set of all integers. Determine whether $R$ is reflexive, symmetric, antisymmetric or transitive where $(x,y) \in R$ iff

(a) $xy > 1$ (b) $|y-x|+2$ is a prime.

Soln (a) $xRy$ iff $xy > 1$

Not reflexive as $(1,1) \notin R$ $(\because 1 \cdot 1 \not> 1)$

Symmetric – Yes $xy > 1 \Rightarrow yx > 1$

transitive – Yes

antisymm – NO $xy > 1$ & $yx > 1 \not\Rightarrow x = y$

$xy > 1$ & $yz > 1 \Rightarrow z > \frac{1}{y}$.

$\rightarrow xz = xy \cdot \frac{z}{y} > \frac{z}{y}$ $(\because xy > 1)$ $> ?$

(b) $(x,y) \in R$ iff $|y-x|+2$ is prime

(1) Reflexive - Yes $|x-x|+2 = 2 \rightarrow$ prime

(2) If $|y-x|+2$ is prime $\Rightarrow |x-y|+2$ is prime
$\Rightarrow R$ is symmetric

(3) transitive: NO,
$(x,y) \in R \Rightarrow |y-x|+2$ is prime
$(y,z) \in R \Rightarrow |z-y|+2$ is prime

$|z-x|+2$
$= |z-y+y-x|+2$
$\neq |z-y| + |y-x| + 2$

(4) $(x,y) \in R \Rightarrow |y-x|+2$ is prime
$(y,x) \in R \Rightarrow |x-y|+2$ is prime $\Big] \not\Rightarrow x=y$.

$\therefore$ Not anti symmetric.

Ex Give an example of relation which is
(a) reflexive but not symmetric & transitive
(b) symmetric but not reflexive and transitive
(c) transitive and reflexive but not symmetric

soln Let $A = \{1,2,3\}$

(a) $R = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$

(b) $R = \{(1,2), (2,1), (3,3)\}$.

(c) $R = \{(1,2), (2,1), (1,1), (2,2), (1,3), (2,3)\}$.

Equivalence Relation: A relation on set $A$ is said to be an equivalence relation if it is
(1) Reflexive (2) Symmetric (3) transitive.

Ex Let $\mathbb{Z}_0$ denote set of all non-zero integers and $R$ be relation on $\mathbb{Z}_0$ defined by $x^y = y^x$ where $x, y \in \mathbb{Z}_0$. Is $R$ an equivalence relation?

$$R = \{(x,y) : x^y = y^x,\ x, y \in \mathbb{Z}_0\}$$

Soln (1) Reflexive

Let $x \in \mathbb{Z}_0$. Then $x^x = x^x \Rightarrow (x,x) \in R \ \forall\ x \in \mathbb{Z}_0$.

$\therefore$ Reflexive.

(2) Symmetric:

Let $(x,y) \in R \Rightarrow x^y = y^x$

$\Rightarrow y^x = x^y \Rightarrow (y,x) \in R$.

Hence, symmetric

(3) Transitive:

Let $(x,y), (y,z) \in R \Rightarrow x^y = y^x$ and $y^z = z^y$.

$\Rightarrow (x^y)^z = (y^x)^z$ and $\Rightarrow (y^z)^x = (z^y)^x$

$\Rightarrow (x^z)^y = (y^z)^x$ — (1) $\Rightarrow (y^z)^x = (z^x)^y$ — (2)

From (1) & (2)

$(x^z)^y = (z^x)^y \Rightarrow x^z = z^x \Rightarrow (x,z) \in R$.

Hence, transitive.

Ex Let $\mathbb{Z}$ be set of integers and $X = \mathbb{Z} \times \mathbb{Z}$. Define relation $\sim$ on $X$ as $(a,b) \sim (c,d)$ if $ad = bc$. Prove that '$\sim$' is an equivalence relation.

Soln (1) Let $(a,b) \in X$. Now, $ab = ba$ $(\because a, b \in \mathbb{Z})$

$\Rightarrow (a,b) \sim (a,b)$

$\Rightarrow \sim$ is reflexive.

(2) Let $(a,b),(c,d) \in X$

Let $(a,b) \sim (c,d) \Rightarrow ad = bc$

$\Rightarrow bc = ad$

$\Rightarrow (c,d) \sim (a,b) \Rightarrow \sim$ is symmetric

(3) Let $(a,b),(c,d),(e,f) \in X$

$(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$

$\Rightarrow ad = bc \qquad \Rightarrow cf = de$

$\Rightarrow \frac{a}{b} = \frac{c}{d} \qquad \Rightarrow \frac{c}{d} = \frac{e}{f}$

$\Rightarrow \frac{a}{b} = \frac{e}{f} \Rightarrow af = be \Rightarrow (a,b) \sim (e,f)$

$\Rightarrow \sim$ is transitive

Hence, '$\sim$' is an equivalence relation

Ex If R and S are equivalence relations on set A then

(1) $R \cap S$ is an equivalence relation

(2) $R \cup S$ may not be an equivalence relation

Soln (1) Let R & S be equivalence relation on set A.

(1) ~~Let~~ $(x,x) \in R$ & $(x,x) \in S \ \forall x \in A$ ($\because$ R & S are eq. rel hence reflexive)

$\Rightarrow (x,x) \in R \cap S \ \forall x \in A$

$\Rightarrow R \cap S$ is reflexive

(2) Let $(x,y) \in R \cap S \Rightarrow (x,y) \in R$ & $(x,y) \in S$

$\Rightarrow (y,x) \in R$ & $(y,x) \in S$ ($\because$ R & S are eq rel. hence symm)

$\Rightarrow (y,x) \in R \cap S$.

$\Rightarrow R \cap S$ is symmetric

(3) Let $(x,y),(y,z) \in R \cap S \Rightarrow (x,y),(y,z) \in R$ and $(x,y),(y,z) \in S$

$\Rightarrow (x,z) \in R$ and $(x,z) \in S$

($\because$ R & S are eq rel hence transitive)

$\Rightarrow (x,z) \in R \cap S \Rightarrow R \cap S$ is transitive

$\Rightarrow R \cap S$ is an eq. relation//

(2) Let $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3), (1,3), (3,1)\}$ ] eq.-rel
$S = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

$R \cup S = \{(1,1), (2,2), (3,3), (1,3), (3,1), (1,2), (2,1)\}$.

$(2,1)$ & $(1,3) \in R \cup S$ But $(2,3) \notin R \cup S$.

$\therefore R \cup S$ may not be eq. relation.

Ex If $R$ is an equivalence relation on $A$ then $R^{-1}$ is also an equivalence relation on $A$.

Soln (1) Let $(x,x) \in R \;\forall x$ ($\because R$ is reflexive)
$\Rightarrow (x,x) \in R^{-1} \;\forall x$
$\Rightarrow R^{-1}$ is reflexive

(2) $(x,y) \in R^{-1} \Rightarrow (y,x) \in R \Rightarrow (x,y) \in R$ ($\because R$ is symm)
$\Rightarrow (y,x) \in R^{-1}$.

$\Rightarrow R^{-1}$ is symmetric

(3) $(x,y), (y,z) \in R^{-1} \Rightarrow (y,x), (z,y) \in R$
$\Rightarrow (z,x) \in R$ ($\because R$ is transitive)
$\Rightarrow (x,z) \in R^{-1}$

$\Rightarrow R^{-1}$ is transitive.

Hence, $R^{-1}$ is an equivalence relation.

Ex Let $R$ be relation on set of integers $\mathbb{Z}$ defined by $a \equiv b \pmod 3$. Show that it is an equivalence relation

Soln $a \equiv b \pmod 3 \Rightarrow a-b$ is a multiple of 3
i.e, $a-b = 3k$, $k \in \mathbb{Z}$.

(1) Reflexive: $x \equiv x \pmod 3$ as $x - x = 0(3)$ $0 \in \mathbb{Z}$.
$\Rightarrow (x,x) \in R \ \forall x \in \mathbb{Z}$. Hence reflexive

(2) Symmetric: $(x,y) \in R \Rightarrow x - y = 3k$, $k \in \mathbb{Z}$.
$\Rightarrow y - x = 3(-k)$, $-k \in \mathbb{Z}$.
$\Rightarrow (y,x) \in R$. Hence symmetric

(3) Transitive: $(x,y),(y,z) \in R \Rightarrow x - y = 3k$, $y - z = 3l$
$k, l \in \mathbb{Z}$
$\therefore x - z = x - y + y - z = 3k + 3l = 3(k+l)$ $k+l \in \mathbb{Z}$.
$\Rightarrow (x,z) \in R$. Hence transitive.

$\Rightarrow a \equiv b \pmod 3$ is an equivalence relation on $\mathbb{Z}$.

Equivalence class: Let R be an equivalence relation in set A. Then set of all elements of R that are related to x ~~underR~~ is called equivalence class of x under R i.e,
$[x] = \{y : y \in A \text{ and } (x,y) \in R\}$.

Note: $A/R = \{[x] : x \in A\} \to$ set of all equivalence classes of elements of A under eq. relation R is called quotient set of A by R