Zeros of Analytic Function

A zero of analytic function $f(z)$ is value of z such that $f(z) = 0$.

If $f(z)$ is analytic in the nbd of a point $z = a$ then by Taylor's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{where } a_n = \frac{f^n(a)}{n!}$$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n$$

If $a_0 = a_1 = \dots = a_{m-1} = 0$ but $a_m \neq 0$

then $f(z)$ is said to have a zero of order m.

The zero is said to be simple if $m = 1$

It may also seen that since $a_m = \frac{f^m(a)}{m!}$

at a zero of order m, we have

$$f(a) = f'(a) = \dots = f^{m-1}(a) = 0 \text{ but } f^m(a) \neq 0$$

Thus in the nbd of the zero $z = a$ of order m the function $f(z)$ is

$$f(z) = a_m (z-a)^m + a_{m+1}(z-a)^{m+1} + \dots$$

$$= (z-a)^m [a_m + a_{m+1}(z-a) + \dots]$$

$$= (z-a)^m \phi(z)$$

where $\phi(z) = a_m + a_{m+1}(z-a) + \dots$

is analytic and non zero at and in the nbd of the zero $z = a$.

SINGULAR POINT

A point z_0 is called a singular point of a function f if f fails to be analytic at z_0 but is analytic at some point in every nbd of z_0.

A singular point z_0 is said to be isolated if, in addition, there is some nbd of z_0 throughout which f is analytic except at the point itself, otherwise it is called non isolated.

For example. ① $f(z) = \dfrac{z+1}{z(z-2)}$ is analytic everywhere except at $z=0$ and $z=2$, which are isolated singular points.

② Consider the function

$$f(z) = \frac{1}{\tan(\pi/z)}$$ This

function is ~~an~~ not analytic at the points where $\tan \pi/z = 0$ i.e at the points $\pi/z = n\pi$ or $z = \frac{1}{n}$ $(1,2,3,\ldots)$. Thus $z = 1, z=1/2, z=1/3$ --- $z=0$ are singularities of the function all of which are isolated except $z=0$ because ~~there is~~ in nbd of $z=0$ there are infinite number of other singularities $z=1/n$

Kinds of Singularities =

We have seen if a function $f(z)$ has an isolated singular point z_0, then f can be represented by Laurent's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$

in a domain $0 < |z-z_0| < R_1$ centred at that point.

The portion of the series involving negative power of $z-z_0$ is called the principal part of f at z_0. We now use the principal part to distinguish between three types of isolated singular points. The behaviour of f near z_0 is fundamentally different in each case.

(1) - Removable Singularity

When all the coeff. b_n in the principal part of f at an isolated singular point z_0 are zero, the point z_0 is called removable singular point of f. In this case Laurent's series contains only non negative powers of $z-z_0$ and the series is infact, a power series. Note that residue at a removable singular point is always zero. If we define $f(z)$ as a_0 at z_0 the function becomes analytic at z_0. Thus a function with a removable singular point can be made analytic at that point by ~~assing~~ assigning a suitable value to the function there.

(Ex.) Consider the function

$$f(z) = \frac{e^z - 1}{z} = \frac{1}{z}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots - 1\right)$$

Alternative def – An singularity $z=a$ is called a removable singularity of $f(z)$ if $\lim_{z\to a} f(z)$ exist finitely.

Example – suppose $f(z) = \frac{\sin z}{z}$ Then

$$\lim_{z\to 0} \frac{\sin z}{z} = 1.$$

Again

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)$$
$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

Since no negative powers occures. Hence z is a removal singularity of $f(z)$.

(II) Pole – If the principal part contains a finite number of terms, say m, then the singularity $z=a$ is called a pole of order m of $f(z)$. Then $f(z)$ will have the expansion of form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{m} b_n (z-a)^{-n}$$

Alternative def. – If there exist a +ve integer s.t.

$$\lim_{z\to a} (z-a)^n f(z) = A$$

$A \neq 0$

Then $z=a$ is called pole of order n.

Example – if $f(z) = \frac{1}{(z-5)^3(z-4)^2}$ then

$z=5$ is pole of order 3 and $z=4$ is pole of order 2.

Essential singularity –

If principal part contains infinite number of terms i.e. if series $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ contains an infinite number of terms ~~i.e the series~~ then singularity $z = a$ is called a essential singularity.

Alternative def. – If there exist no finite value of n s.t.

$$\lim_{z \to a} (z-a)^n f(z) = c \quad \text{finite \& non zero const.}$$

then $z = a$ is called an essential singularity

Example $z = 0$ is an essential singularity of $e^{1/z}$ since the expansions of $e^{1/z}$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{z^3 3!} + \cdots$$

is an infinite series of negative powers of z.

Theorem ① If $f(z)$ has a pole at $z = a$ then

$$|f(z)| \to \infty \text{ as } z \to a$$

Suppose $f(z)$ has a pole of order m at $z = a$

To prove that

$$|f(z)| \to \infty \text{ as } z \to a$$

By assumption principal parts contains only m terms so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{m} b_n (z-a)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{1}{(z-a)^m}\left[(z-a)^{m-1} b_1 + (z-a)^{m-2} b_2 + \cdots + b_m\right]$$

The expression within square bracket on R.H.S $\to \infty$ as $z \to a$ so that the whole R.H.S expression $\to \infty$ as $z \to a$

consequently $|f(z)| \to \infty$ as $z \to a$

Theorem (2) If an analytic function $f(z)$ has a pole of order m at $z = a$. then $\frac{1}{f(z)}$ has a zero of order m at $z = a$ and conversely.

Proof. (T. 2/233)

Theorem (3) **Zeros are isolated -**

Proof- let $z = a$ be zero of order m of an analytic function $f(z)$. then we may write $f(z) = (z-a)^m \phi(z)$

where $\phi(z)$ is analytic and $\phi(a) \neq 0$

Evedently $(z-a)^m \neq 0$ at $z \neq a$

Now there exist no other point in the deleted nbd $|z-a| < r$ at which $f(z) = 0$

Hence the zero $z = a$ is an isolated singularity this is true for every zero of $f(z)$. Therefore the zeros of $f(z)$ are isolated.

Theorem (4) **Poles are isolated -**

Proof-

Let $z = a$ be a pole of order m of the an analytic function $f(z)$, then $\frac{1}{f(z)}$ is an analytic and has a zero of order m at $z = a$. since zeros are isolated and hence

Solved Examples.

Ex.1.

Find the singularities of the function $\dfrac{e^{c/(z-a)}}{e^{z/a}-1}$, indicating the character of each singularity.

$$\text{Let } f(z) = \frac{e^{\frac{c}{z-a}}}{e^{z/a}-1}$$

$$= \frac{\exp\left(\frac{c}{z-a}\right)}{\exp(1)\,\exp\left(\frac{z-a}{a}\right)-1}$$

$$= \frac{e^{c/(z-a)}}{e\,e^{(z-a)/a}-1}$$

$$= -e^{c/(z-a)}\left[1-e.e^{(z-a)/a}\right]^{-1}$$

$$= -e^{c/(z-a)}\cdot\left[1-e\left\{1+\frac{z-a}{a}+\left(\frac{z-a}{a}\right)^2\frac{1}{2!}+\cdots\right\}\right]^{-1}$$

$$= -\left[1+\frac{c}{(z-a)}+\left(\frac{c}{z-a}\right)^2\cdot\frac{1}{2!}+\cdots\right]$$

$$\times\left[1+e\left\{1+\frac{z-a}{a}+\left(\frac{z-a}{a}\right)^2\cdot\frac{1}{2!}+\cdots\right\}\right.$$

$$\left.+e^2\left\{1+\left(\frac{z-a}{a}\right)+\cdots\right\}^2+\cdots\right]$$

Clearly this expansion contain positive and negative powers of $z-a$. In particular, term containing negative powers of $z-a$ are infinite in number. Hence by def. $z=a$ is an essential singularity.

(ii)

$$f(z) = \frac{\exp\left(\frac{c}{z-a}\right)}{\exp\left(\frac{z}{a}\right)-1}$$

Evidently denominator has zero of order 1 at

$$e^{z/a} = 1 = e^{2\pi n i}$$

i.e

$$z = 2\pi n i$$

consequently $f(z)$ has a pole of order one at each point $z = 2n\pi i a$ (where $n = 0, \pm 1, \pm 2 \cdots$)

Ex.2

Show that the function e^z has an isolated essential singularity at $z=\infty$

Soln. Let $f(z) = e^z$ — (1)

The behaviour of $f(z)$ at $z=\infty$ is the same as the behaviour of $f(\frac{1}{z})$ at $z=0$

$$(1) \Rightarrow f(\tfrac{1}{z}) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2}\cdot\frac{1}{2!} + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{z^n n!}$$

or $$f(\tfrac{1}{z}) = 1 + \sum_{n=1}^{\infty} \frac{1}{(z-0)^n n!}$$

This is Laurent's expansion of $f(\frac{1}{z})$ about the the point $z=0$. This expansion contains an infinite number of terms in the negative power of z. Hence by def. $z=0$ is an essential singularity of $f(\frac{1}{z})$. Consequently $f(z)$ has essential singularity at $z=\infty$.

Imp Example. (1) The origin is a singular point of $\log z$, but it is not an isolated singular point since every nbd of the origin contains points on the negative real axis and $\log z$ fails to be analytic at each of those points.

(2) The function $\frac{1}{\sin \pi/z}$ has the singular points $z=0$ and $z=\frac{1}{n}$ ($n = \pm 1, \pm 2$ ---) all lying on the segment of the real axis from $z=-1$ to $z=1$. Each singular point except $z=0$ is isolated. The singular point