Date
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Unit - II
Linear Vector Space

Vector space - Let $V = R^n$ where R^n consists of all n-element sequences $u = (a_1, a_2, ---a_n)$ of real numbers called the components of u.
The term vector is used for the elements of V and we denote them using the letters u, v and w, with or without a subscript. The real numbers we call scalars and we denote them using letters other than u, v or w.

Here two operations on $V = R^n$.

(a) **Vector Addition** - Given vectors $u = (a_1, a_2, ---, a_n)$ and $v = (b_1, b_2, ---, b_n)$ in V, we define the vector sum u+v by

$$u+v = (a_1+b_1, a_2+b_2, ---, a_n+b_n)$$

for this, we add corresponding components of the vectors.

(b) **Scalar Multiplication** -

Given a vector $u = (a_1, a_2, ---a_n)$ and a scalar k in R, we define the scalar product ku by

$$ku = (ka_1, ka_2, ---, ka_n)$$

That is, we multiply each component of u by the scalar k.

Subspaces - If V is a vector space over a field F and U is a subset of V. An operation on U is a function $U \times U \rightarrow U$ it does so if and only if adding two elements of U always gives another element of U.

A subset U of a vector space V is said to be closed under addition and scalar multiplication if,

(i) $u_1 + u_2 \in U$ for all $u_1, u_2 \in U$

(ii) $\lambda u \in U$ for all $u \in U$ and all scalars λ

A subset U of a vector space V is called a subspace of V if U is itself a vector space relative to addition and scalar multiplication inherited from V.

Theorem-1. If V is a vector space and U a subset of V which is nonempty and closed under addition and scalar multiplication, then U is a subspace of V.

Proof → Let $x, y, z \in U$.

then $x, y, z \in V$

for V it follows that

$(x+y)+z = x+(y+z)$

and this also holds in U.

Let $x, y \in U$. Then $x, y \in V$.

so $x+y = y+x$.

Now we would prove that U has a zero element.

Since V is a vector space we know that V has a zero element, which

we will denote by '0'.
Since U is nonempty there certainly exists at least one element in U. Let x be such an element.
By closure under scalar multiplication we have $0x \in U$.
$0x = \underset{\sim}{0}$ (since x is an element of V).
So it is necessarily true that $\underset{\sim}{0} \in U$
It is trivial that $\underset{\sim}{0}$ is also a zero element for U, if $y \in U$ is arbitrary, then $y \in V$, it gives $\underset{\sim}{0} + y = y$.

$x \in U$ has a negative in U.
Since V guarantees that x has a negative in V and since the zero of U is the same as the zero of V.
But $x \in U$ it suffices that $-x \in U$ which is given $(-1)x \in U$ closure under scalar multiplication.

$-x = (-1)x$