The wave equation for $\vec{E}$ and $\vec{B}$ →

In vacuum where there is no charge or current, we can write Maxwell's equations as,

(i) $\nabla \cdot \vec{E} = 0$

(ii) $\nabla \cdot \vec{B} = 0$

(iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(iv) $\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Now taking equation (iii).

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 E$$

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times -\left(\frac{\partial \vec{B}}{\partial t}\right)$$

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\nabla \times (\nabla \times \vec{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

Now taking equation (iv)

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot B) - (\nabla^2 B)$$

$$\nabla \times (\nabla \times \vec{B}) = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\nabla \times (\nabla \times \vec{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \vec{E})$$

$$\nabla \times (\nabla \times \vec{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

Since we have taken $\nabla \cdot E = 0$ in (i) and $\nabla \cdot \vec{B} = 0$ in (ii).

So $$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 B = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

In vacuum, each cartesian component of $\vec{E}$ and $\vec{B}$ satisfies the three dimensional wave equation. $\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

Maxwell's equations imply that empty space supports the propagation of electro-magnetic waves travelling at a speed,

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3\times10^8 \text{ m/sec} = \text{velocity of light} = c$$

Monochromatic Plane Waves →

We can confine our attention to sinusoidal waves of frequency ω. Since different frequencies in the visible range corresponds to different colors, these waves are called monochromatic. The waves are traveling in the z direction and have no x or y dependence, are called plane waves, because the fields are uniform over every plane perpendicular to the direction of propagation.

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}$$

$$\tilde{B}(z,t) = \tilde{B}_0 e^{i(kz-\omega t)}$$

Fields amplitude

The wave eq. for $\vec{E}$ and $\vec{B}$ were derived from Maxwell's equations. Every solution to Maxwell's equations must obey the wave equation. the converse is not true.

In particular case, when we are taking vacuum then $\nabla \cdot \vec{E} = 0$ and $\nabla \cdot \vec{B} = 0$

It follows that -

$$(\tilde{E}_0)_z = (\tilde{B}_0)_z = 0$$

that is, the electromagnetic waves are transverse, the electric and magnetic fields are perpendicular to the direction of propagation

Faraday's law $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

it implies a relation between the electric and magnetic amplitudes,

$$\left.\begin{aligned} -k\,(\tilde{E}_0)_y &= \omega\,(\tilde{B}_0)_x \\ k\,(\tilde{E}_0)_x &= \omega\,(\tilde{B}_0)_y \end{aligned}\right] - 1(A)$$

$$\tilde{B}_0 = \frac{k}{\omega}(\hat{z} \times \tilde{E}_0) \quad - 1(B)$$

evidently $\vec{E}$ and $\vec{B}$ are in phase and mutually perpendicular, their real amplitudes are related by,

$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0$$

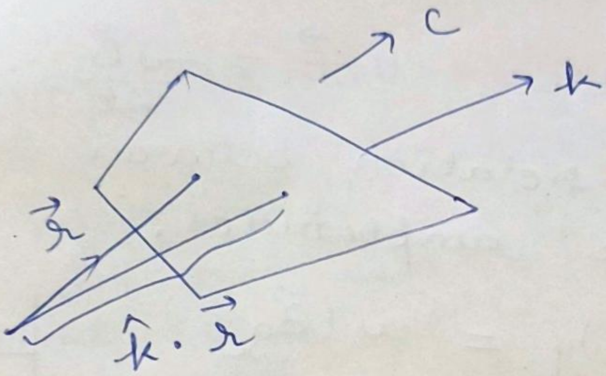
the propagation vector $\vec{k}$ pointing in the direction of propagation, whose magnitude is the wave vector k.

$k \cdot r$ is the generalization of kz

so,

$$\boxed{\begin{aligned} \tilde{E}(r,t) &= \tilde{E}_0 e^{i(k\cdot r - \omega t)}\hat{n} \\ \tilde{B}(r,t) &= \frac{1}{c}\tilde{E}_0 e^{i(k\cdot r - \omega t)}(\hat{k}\times\hat{n}) \\ &= \frac{1}{c}\hat{k}\times\tilde{E} \end{aligned}}$$

$\hat{n}$ is the palarization vector.
$\vec{E}$ is transverse so $\boxed{\hat{n}\cdot\hat{k} = 0}$



the (real) electric and magnetic fields in a monochromatic plane with propa-gation vector $\vec{k}$ and palarization $\hat{n}$ are,

$$\vec{E}(\vec{r},t) = E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)\hat{n} \quad \text{— 2(A)}$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)(\hat{k}\times\hat{n}) \quad \text{— 2(B)}$$